Graph Theory

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CONNECTIVITY

G_1 is a tree, a minimal connected graph; deleting any edge disconnects it. G_2 cannot be disconnected by the deletion of a single edge, but can be disconnected by the deletion of one vertex, its cut vertex. There are no cut edges or cut vertices in G_3, but even so G_3 is clearly not as well connected as G_4, the complete graph on five vertices. Thus, intuitively, each successive graph is more strongly connected than the previous one. We shall now define two parameters of a graph, its connectivity and edge connectivity, which measure the extent to which it is connected.

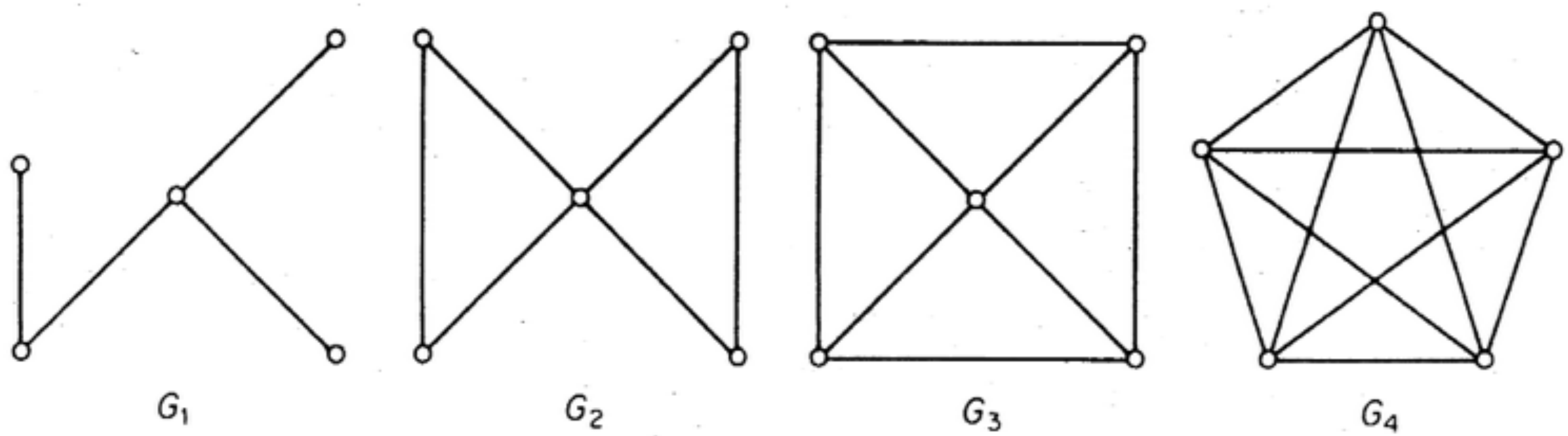

Figure 3.1

A *vertex cut* of G is a subset V' of V such that $G-V'$ is disconnected.

A *k-vertex cut* is a vertex cut of k elements.

A complete graph has no vertex cut; in fact, the only graphs which do not have vertex cuts are those that contain complete graphs as spanning subgraphs.

If G has at least one pair of distinct nonadjacent vertices the *connectivity* $\kappa(G)$ of G is the minimum k for which G has a k-vertex cut; otherwise, we define $\kappa(G)$ to be $\nu-1$.

Conectividade: $\kappa(G)$

- Minimum number of vertices necessary to disconnect G
- If G is complete, we define $\kappa(G)=n-1$
- If G is trivial or disconnected, $\kappa(G)=0$

G is k-connected if $\kappa(G)\geq k$

All nontrivial connected graphs are 1-connected

Recall that an edge cut of G is a subset of E of the form $[S, \bar{S}]$, where S is a nonempty proper subset of V. A *k-edge cut* is an edge cut of k elements. If G is nontrivial and E' is an edge cut of G, then $G-E'$ is disconnected; we then define the *edge connectivity* $\kappa'(G)$ of G to be the minimum k for which G has a k-edge cut. If G is trivial, $\kappa'(G)$ is defined to be zero.

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Edge connectivity: $\kappa'(G)$

- If G is trivial or disconnected, $\kappa'(G)=0$
- G is said to be k-edge connected if $\kappa'(G) \geq k$

All nontrivial connected graphs are 1-edge-connected.

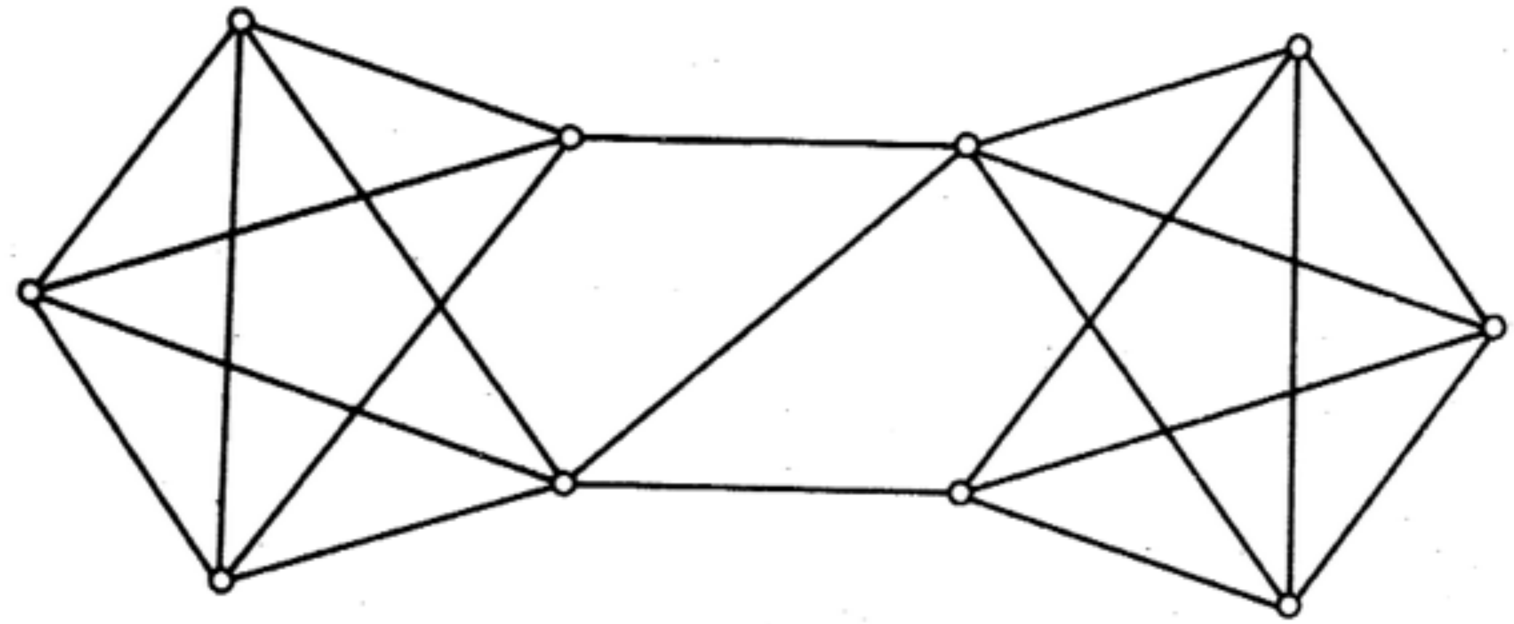

Figure 3.2

Theorem 3.1 $\kappa \leq \kappa' \leq \delta.$

- If G is trivial, $\kappa'=0 \leq \delta$.
- Otherwise, the set of links incidents to the vertex of degree δ constitute a δ–edge-cut of G
  - $\kappa'=0 \leq \delta$

- We prove that $\kappa \leq \kappa'$ by induction on κ'
- If $\kappa'=0$, the result is true, since in this case G is trivial or disconnected
- Suppose the result is true for all graphs with edge connectivity less than k
- Let G be a graph with $\kappa'(G) = k > 0$
- Let e be an edge in a k-edge cut of G.
- Setting, H=G-e,
  - $\kappa'(H)=k-1$

By induction

$\kappa(H) \leq k-1$

- If H contains a complete graph as a spanning subgraph

G does also contain

and $\kappa(G) = \kappa(H) \leq k\text{-}1$

Ohterwise

- S: vertes cut of H with $\kappa(H)$ elements
- Since H-S is disconnected,
  - G-S is disconnected: $\kappa(G) \leq \kappa(H) \leq k\text{-}1$
  - G-S eh conexo

e is a cut edge of G-S

G-S is connected and e is a cut edge:

- $n(G\text{-}S)=2$

$\kappa(G) \le n(G)-1 = \kappa(H)+1 \le k$

- G-S has a 1-vertex cut {v}, implying that $S \cup \{v\}$ is a vertex cut of G and

$\kappa(G) \le \kappa(H)+1 \le k$

Thus in each case we have $\kappa(G) \le k = \kappa'(G)$. The result follows by the principle of induction □

BLOCKS

A connected graph that has no cut vertices is called a *block*. Every block with at least three vertices is 2-connected. A *block of a graph* is a subgraph that is a block and is maximal with respect to this property. Every graph is the union of its blocks; this is illustrated in figure 3.3.

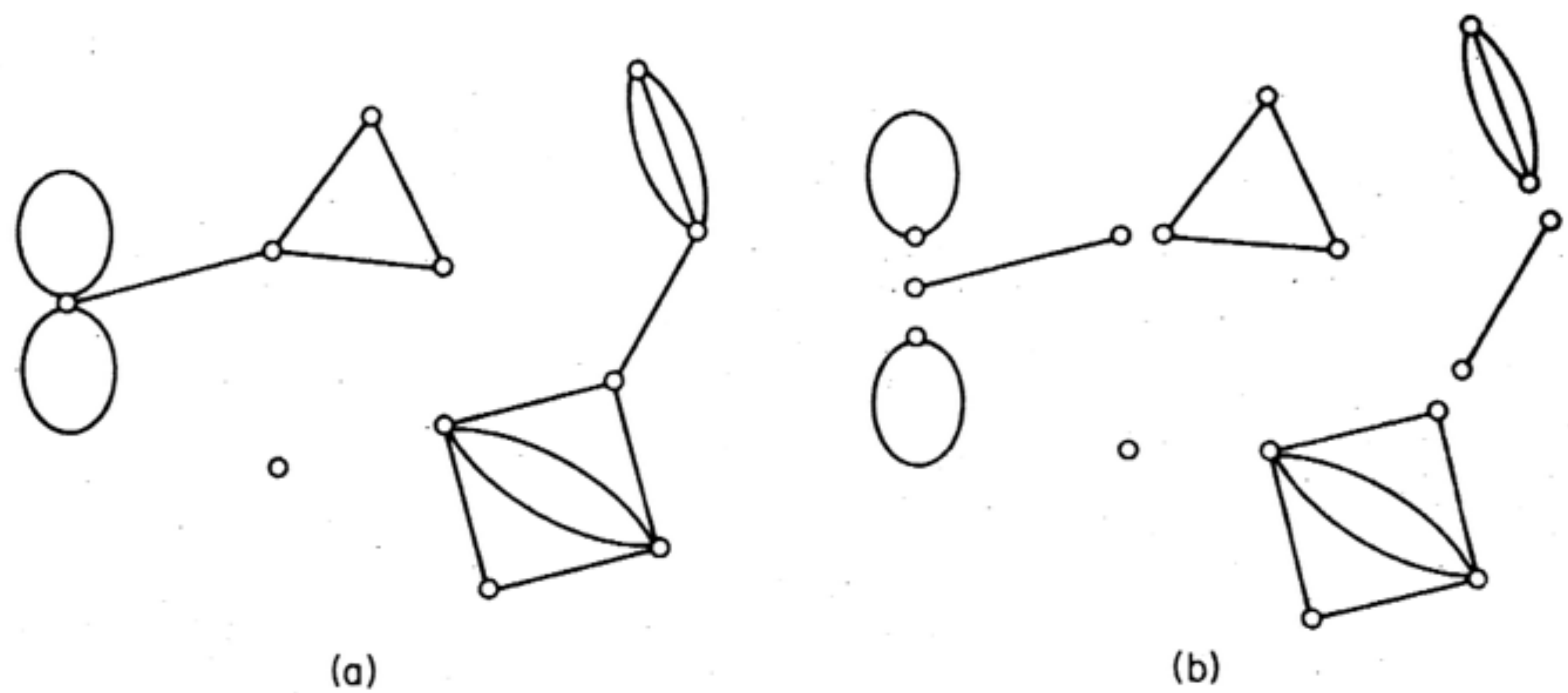

Figure 3.3. (*a*) G; (*b*) the blocks of G

A family of paths in G is said to be *internally-disjoint* if no vertex of G is an internal vertex of more than one path of the family. The following theorem is due to Whitney (1932).

Theorem 3.2 A graph G with $\nu \geq 3$ is 2-connected if and only if any two vertices of G are connected by at least two internally-disjoint paths.

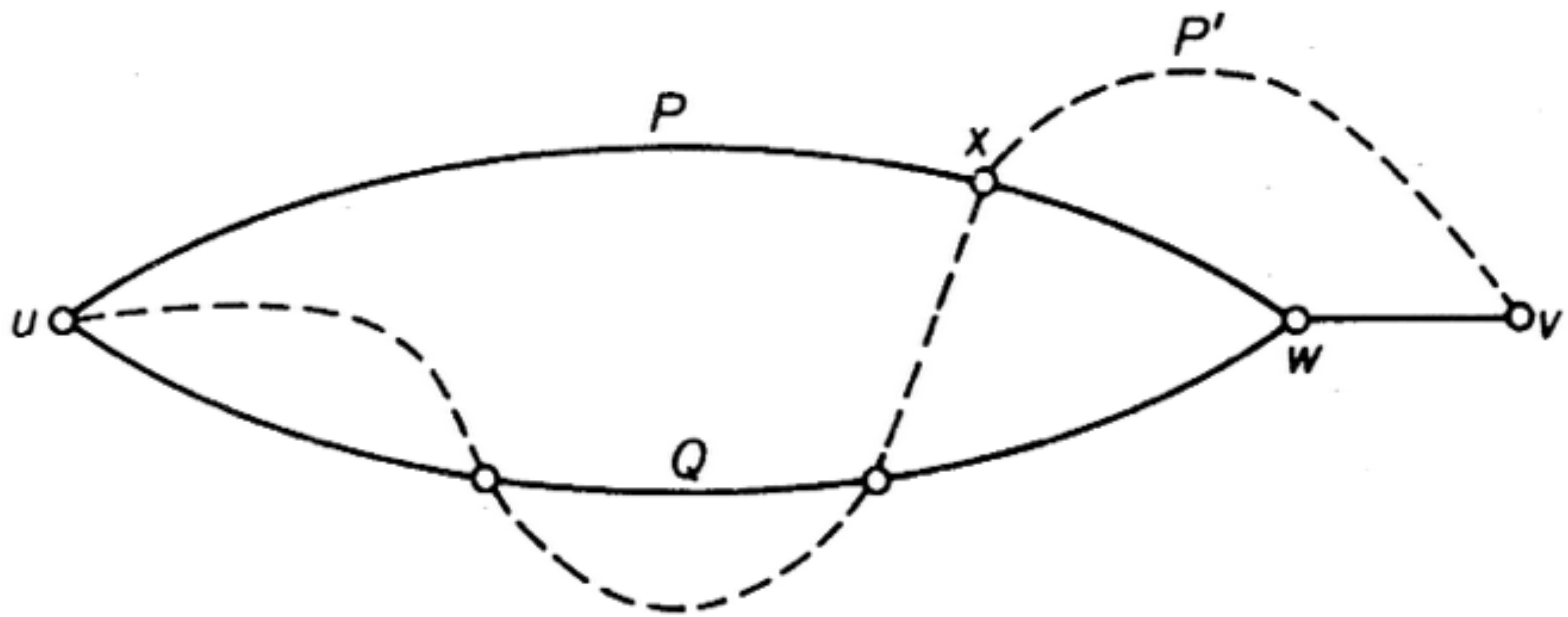

Figure 3.4

Proof If any two vertices of G are connected by at least two internally-disjoint paths then, clearly, G is connected and has no 1-vertex cut. Hence G is 2-connected.

Conversely, let G be a 2-connected graph. We shall prove, by induction on the distance $d(u, v)$ between u and v, that any two vertices u and v are connected by at least two internally-disjoint paths.

Suppose, first, that $d(u, v) = 1$. Then, since G is 2-connected, the edge uv is not a cut edge and therefore, by theorem 2.3, it is contained in a cycle. It follows that u and v are connected by two internally-disjoint paths in G.

Now assume that the theorem holds for any two vertices at distance less than k, and let $d(u, v) = k \geq 2$. Consider a (u, v)-path of length k, and let w be the vertex that precedes v on this path. Since $d(u, w) = k - 1$, it follows from the induction hypothesis that there are two internally-disjoint (u, w)-paths P and Q in G. Also, since G is 2-connected, $G - w$ is connected and so contains a (u, v)-path P'. Let x be the last vertex of P' that is also in $P \cup Q$ (see figure 3.4). Since u is in $P \cup Q$, there is such an x; we do not exclude the possibility that $x = v$.

We may assume, without loss of generality, that x is in P. Then G has two internally-disjoint (u, v)-paths, one composed of the section of P from u to x together with the section of P' from x to v, and the other composed of Q together with the path wv $\square$