Graph Theory

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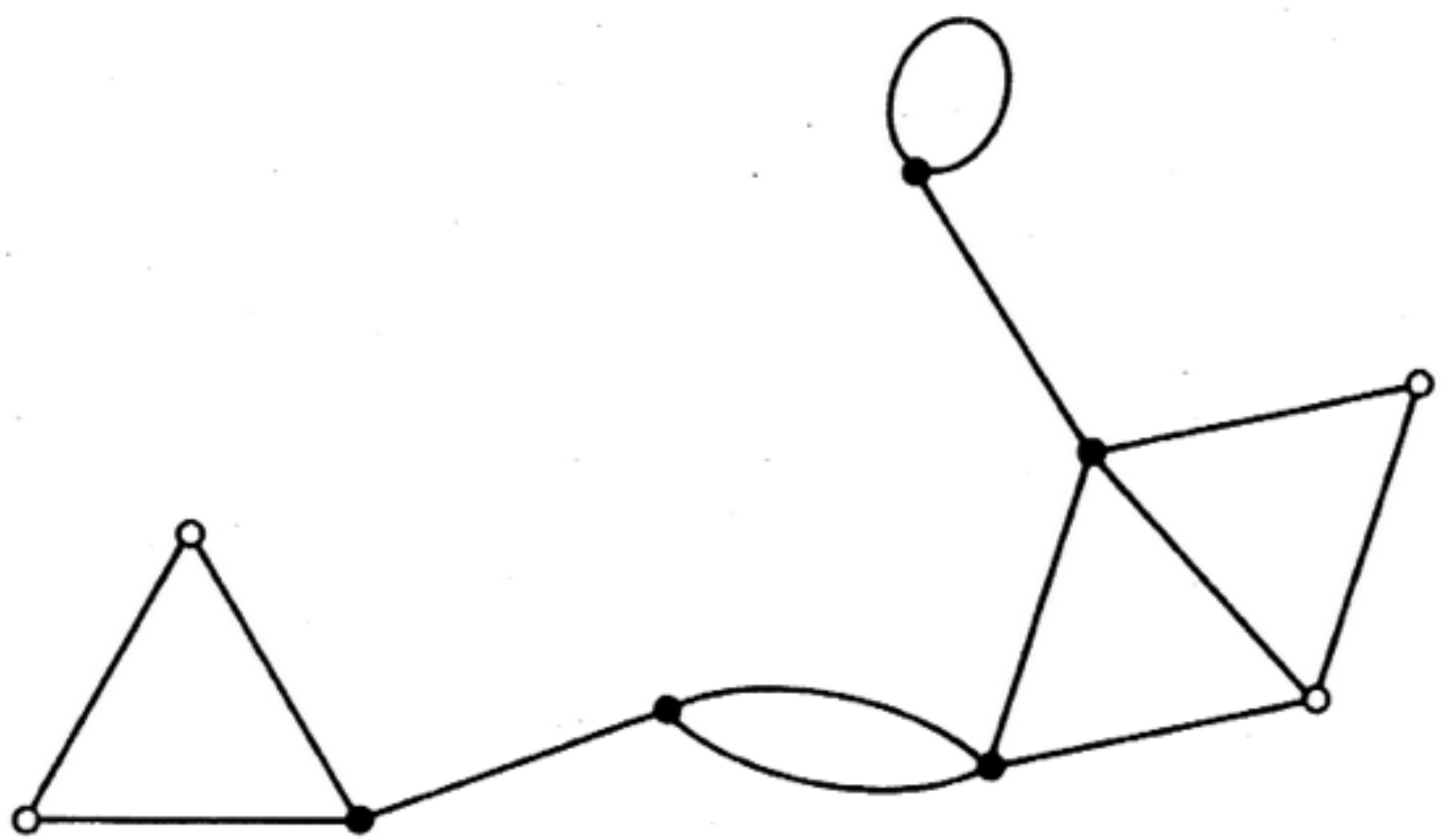
CUT VERTICES

A vertex v of G is a *cut vertex* if E can be partitioned into two nonempty subsets E_1 and E_2 such that $G[E_1]$ and $G[E_2]$ have just the vertex v in common. If G is loopless and nontrivial, then v is a cut vertex of G if and only if $\omega(G-v) > \omega(G)$. The graph of figure 2.5 has the five cut vertices indicated.



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If $d(v)=1$, $G-v$ is an acyclic graph with $\nu(G-v)-1$ edges, and thus (exercise 2.1.5) a tree. Hence $\omega(G-v)=1=\omega(G)$, and v is not a cut vertex of G.

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$$\omega(T-v)=1$$

Since T is a spanning subgraph of G, $T-v$ is a spanning subgraph of $G-v$ and therefore

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It follows that $\omega(G-v)=1$, and hence that v is not a cut vertex of G. Since there are at least two such vertices v, the proof is complete □

Exercises

2.3.1 Let G be connected with $\nu \geq 3$. Show that

(*a*) if G has a cut edge, then G has a vertex v such that $\omega(G-v) > \omega(G)$;

(*b*) the converse of (*a*) is not necessarily true.