Graph Theory

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CUT EDGES AND BONDS

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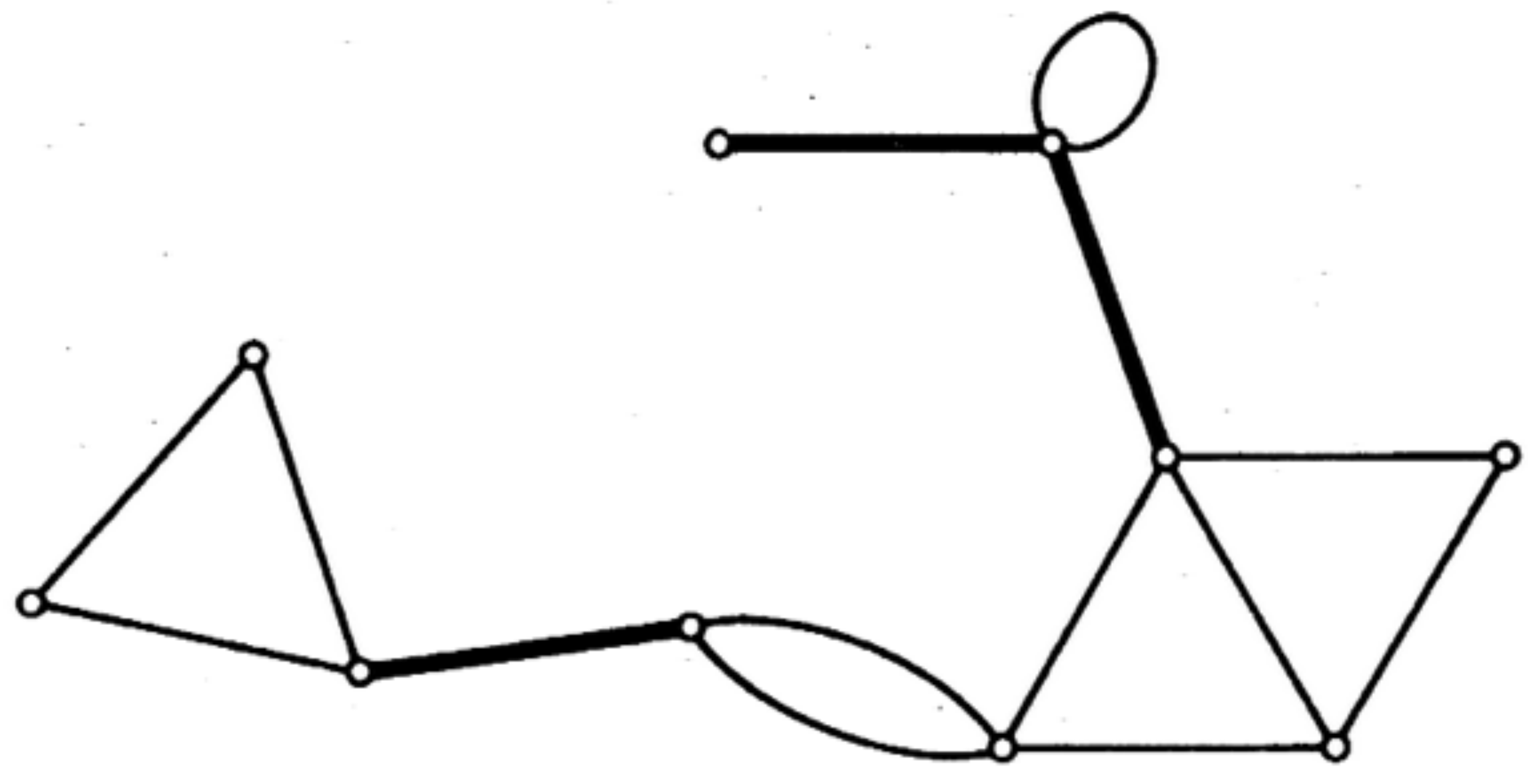

Figure 2.2. The cut edges of a graph

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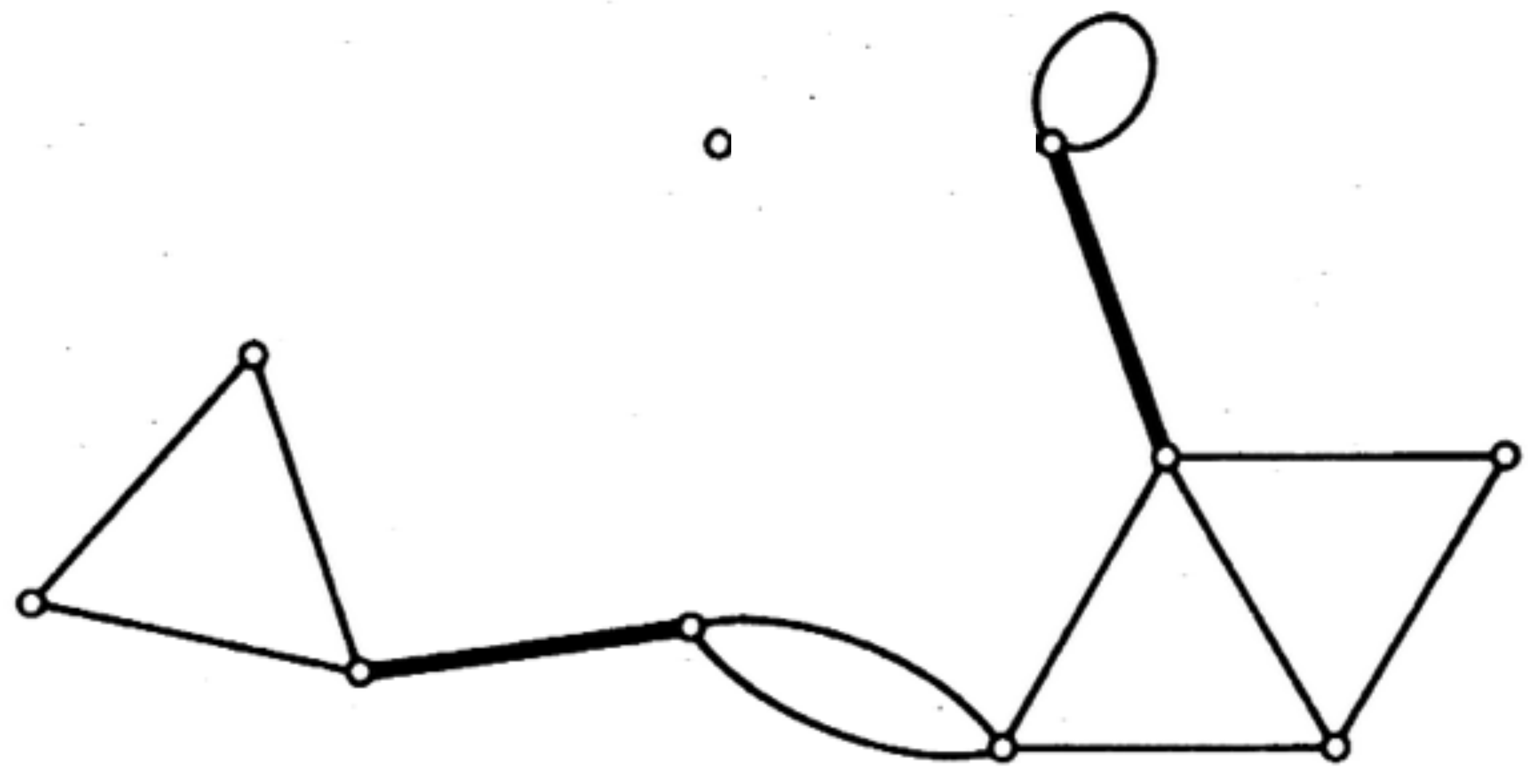

Figure 2.2. The cut edges of a graph

Theorem 2.3 An edge e of G is a cut edge of G if and only if e is contained in no cycle of G.

Proof Let e be a cut edge of G. Since $\omega(G-e)>\omega(G)$, there exist vertices u and v of G that are connected in G but not in $G-e$. There is therefore some (u, v)-path P in G which, necessarily, traverses e. Suppose that x and y are the ends of e, and that x precedes y on P. In $G-e$, u is connected to x by a section of P and y is connected to v by a section of P. If e were in a cycle C, x and y would be connected in $G-e$ by the path $C-e$. Thus, u and v would be connected in $G-e$, a contradiction.

Conversely, suppose that $e = xy$ is not a cut edge of G; thus, $\omega(G-e) = \omega(G)$. Since there is an (x, y)-path (namely xy) in G, x and y are in the same component of G. It follows that x and y are in the same component of $G-e$, and hence that there is an (x, y)-path P in $G-e$. But then e is in the cycle $P+e$ of G □

Theorem 2.4 A connected graph is a tree if and only if every edge is a cut edge.

Proof Let G be a tree and let e be an edge of G. Since G is acyclic, e is contained in no cycle of G and is therefore, by theorem 2.3, a cut edge of G.

Conversely, suppose that G is connected but is not a tree. Then G contains a cycle C. By theorem 2.3, no edge of C can be a cut edge of G □

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Corollary 2.4.1 Every connected graph contains a spanning tree.

Proof Let G be connected and let T be a minimal connected spanning subgraph of G. By definition $\omega(T) = 1$ and $\omega(T - e) > 1$ for each edge e of T. It follows that each edge of T is a cut edge and therefore, by theorem 2.4, that T, being connected, is a tree □

Figure 2.3 depicts a connected graph and one of its spanning trees.

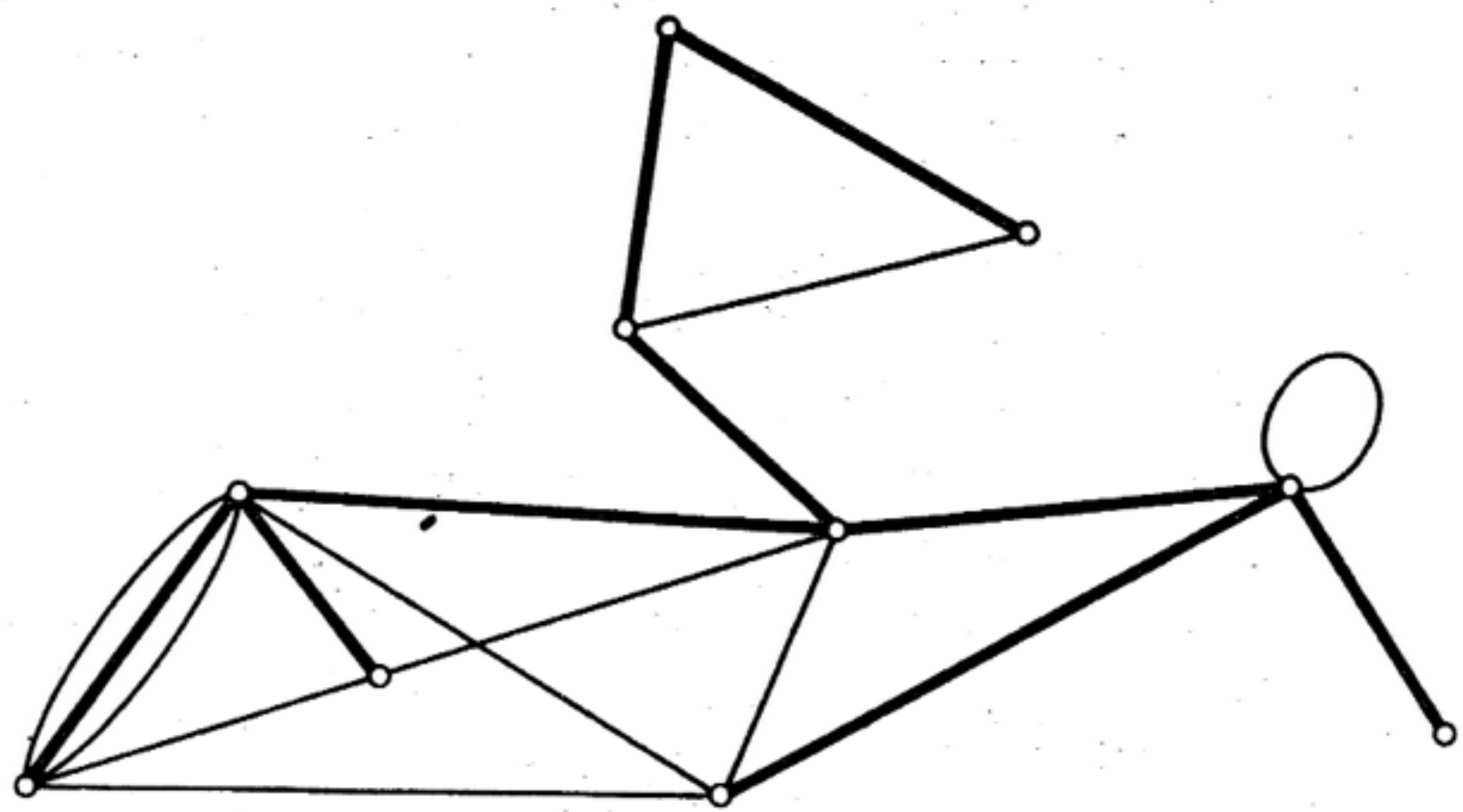

Figure 2.3. A spanning tree in a connected graph

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Theorem 2.5 Let T be a spanning tree of a connected graph G and let e be an edge of G not in T. Then $T+e$ contains a unique cycle.

Proof Since T is acyclic, each cycle of $T+e$ contains e. Moreover, C is a cycle of $T+e$ if and only if $C-e$ is a path in T connecting the ends of e. By theorem 2.1, T has a unique such path; therefore $T+e$ contains a unique cycle $\square$

For subsets S and S' of V, we denote by $[S, S']$ the set of edges with one end in S and the other in S'. An *edge cut* of G is a subset of E of the form $[S, \bar{S}]$, where S is a nonempty proper subset of V and $\bar{S} = V \backslash S$. A minimal nonempty edge cut of G is called a *bond*; each cut edge e, for instance, gives rise to a bond $\{e\}$. If G is connected, then a bond B of G is a minimal subset of E such that $G - B$ is disconnected. Figure 2.4 indicates an edge cut and a bond in a graph.

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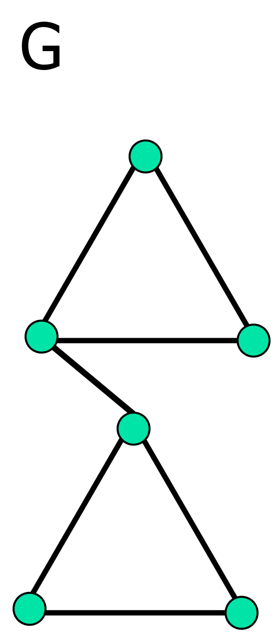

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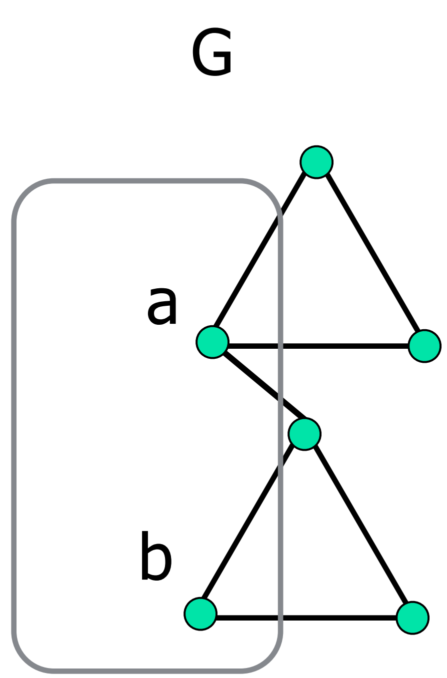

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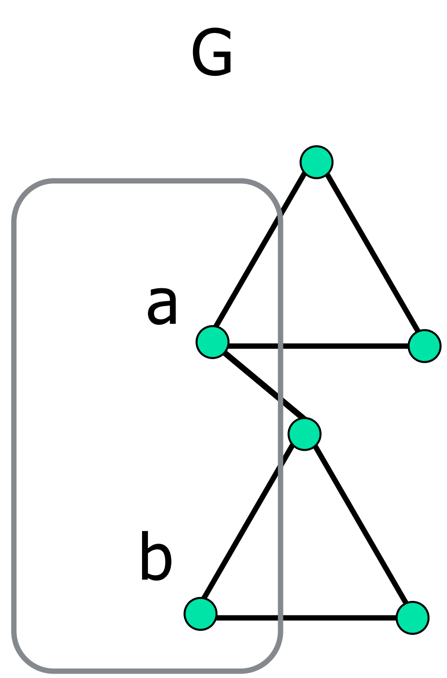

An edge cut

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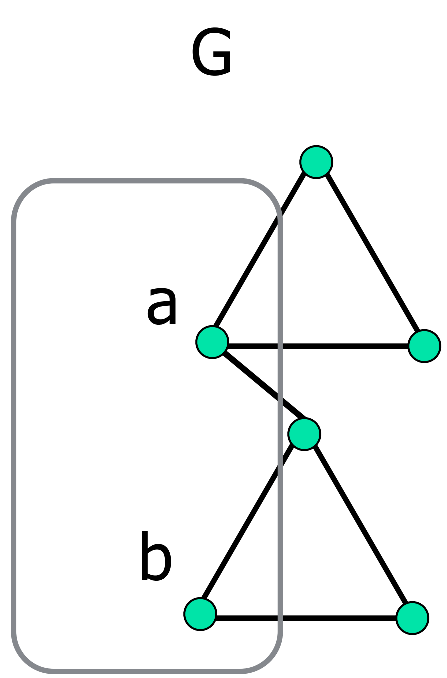

An edge cut, but not a bond
(not minimal)

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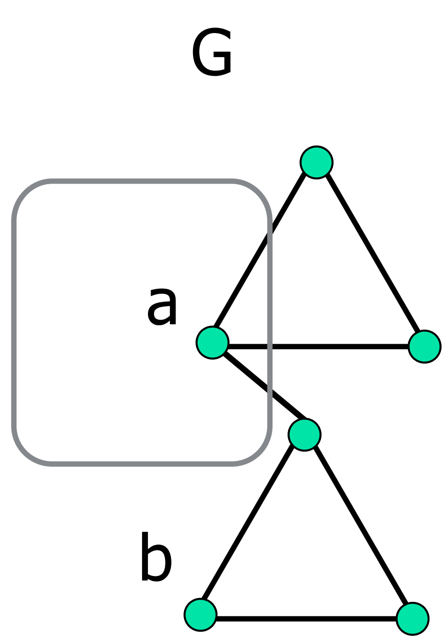

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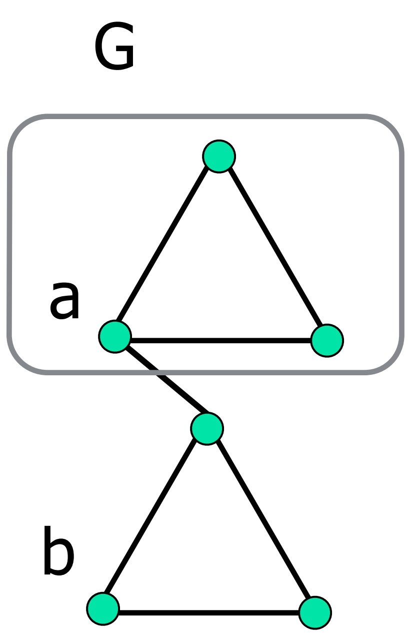

A bond!! (minimal)

CUT VERTICES

A vertex v of G is a *cut vertex* if E can be partitioned into two nonempty subsets E_1 and E_2 such that $G[E_1]$ and $G[E_2]$ have just the vertex v in common. If G is loopless and nontrivial, then v is a cut vertex of G if and only if $\omega(G-v)>\omega(G)$. The graph of figure 2.5 has the five cut vertices indicated.

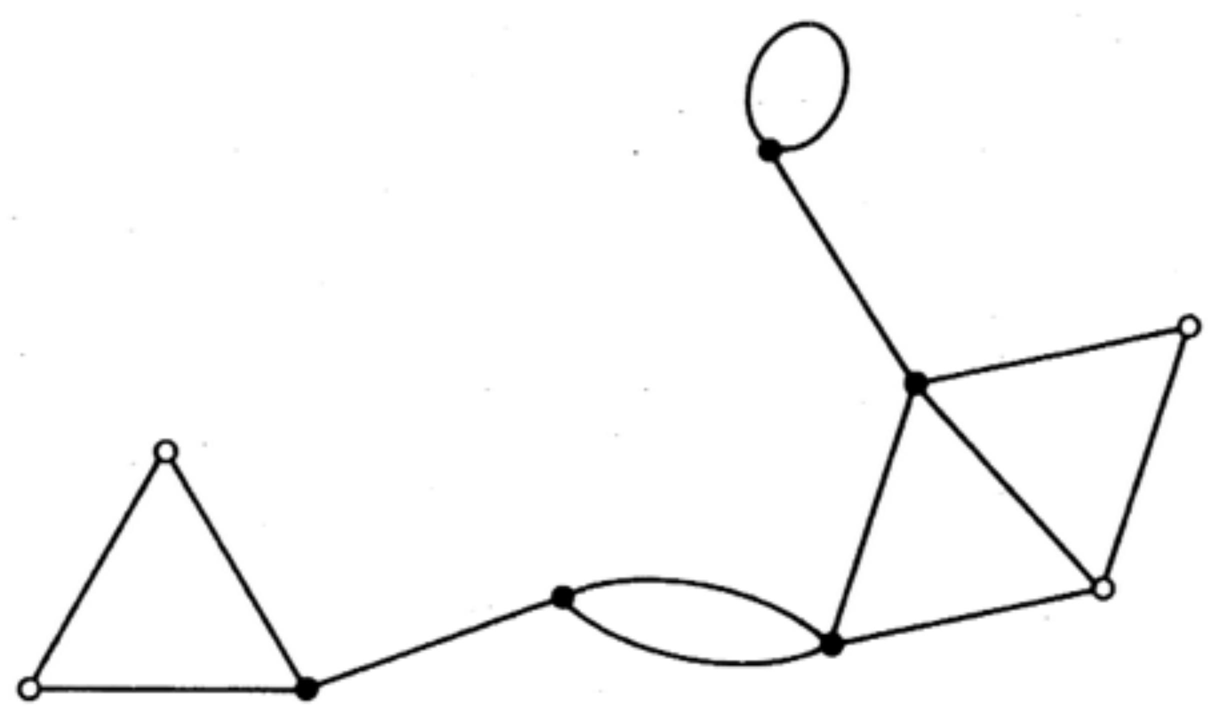

Figure 2.5. The cut vertices of a graph