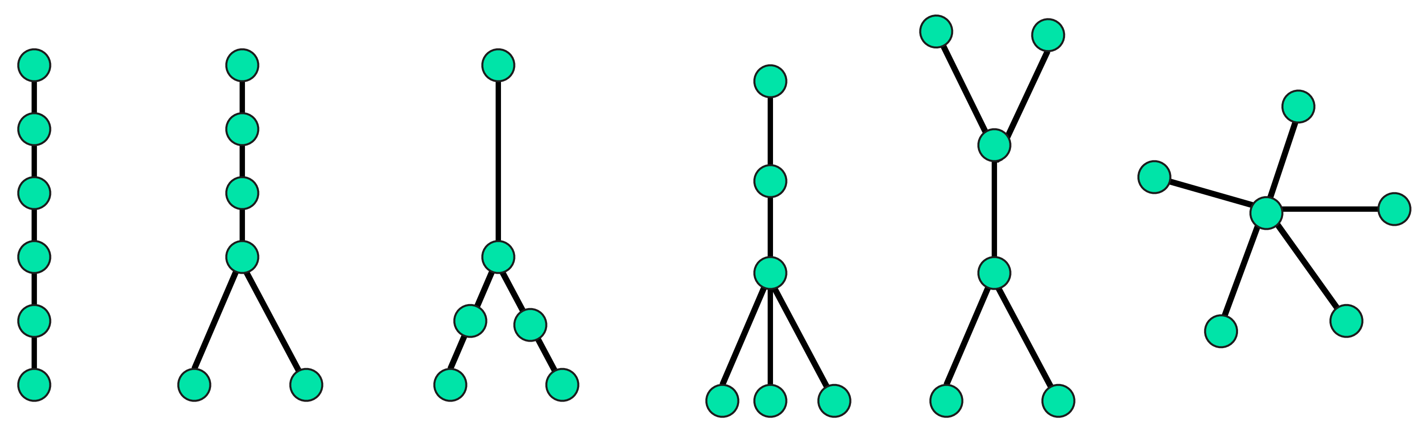
Graph Theory

Loana T. Nogueira

Trees

An *acyclic* graph is one that contains no cycles. A *tree* is a connected acyclic graph. The trees on six vertices are shown in figure 2.1.



Todas as árvores com 6 vértices

Theorem 2.1 In a tree, any two vertices are connected by a unique path.

Theorem 2.1 In a tree, any two vertices are connected by a unique path.

- By contradiction!!!
  - G is a tree
  - P_1 and P_2: two distinct (u,v)-paths
    - There exists e= (x,y) $\in P_1$ such that (x,y) $\notin P_2$
    - The graph ($P_1 \cup P_2$)-e is connected
      - It contains an (x,y)-path
      - Then, P+e is a cycle in G!!

Theorem 2.1 In a tree, any two vertices are connected by a unique path.

- By contradiction!!!
  - G is a tree
  - P_1 and P_2: two distinct (u,v)-paths
    - There exists e= (x,y) $\in P_1$ such that (x,y) $\notin P_2$
    - The graph $(P_1 \cup P_2)$-e is connected
      - It contains an (x,y)-path
      - Then, P+e is a cycle in G!!

The converse of this theorem holds for graphs without loops (exercise

Observe that all the trees on six vertices (figure 2.1) have five edges. In general we have:

Theorem 2.2 If G is a tree, then $\varepsilon = \nu - 1$.

Observe that all the trees on six vertices (figure 2.1) have five edges. In general we have:

Theorem 2.2 If G is a tree, then $\varepsilon = \nu - 1$.

Proof By induction on ν. When $\nu = 1$, $G \cong K_1$ and $\varepsilon = 0 = \nu - 1$.

Suppose the theorem true for all trees on fewer than ν vertices, and let G be a tree on $\nu \geq 2$ vertices. Let $uv \in E$. Then $G - uv$ contains no (u, v)-path, since uv is the unique (u, v)-path in G. Thus $G - uv$ is disconnected and so (exercise 1.6.8a) $\omega(G - uv) = 2$. The components G_1 and G_2 of $G - uv$, being acyclic, are trees. Moreover, each has fewer than ν vertices. Therefore, by the induction hypothesis

$$\varepsilon(G_i) = \nu(G_i) - 1 \quad \text{for} \quad i = 1, 2$$

Thus

$$\varepsilon(G) = \varepsilon(G_1) + \varepsilon(G_2) + 1 = \nu(G_1) + \nu(G_2) - 1 = \nu(G) - 1 \quad \square$$

Corollary 2.2 Every nontrivial tree has at least two vertices of degree one.

Corollary 2.2 Every nontrivial tree has at least two vertices of degree one.

Proof Let G be a nontrivial tree. Then

$$d(v) \geq 1 \quad \text{for all} \quad v \in V$$

Also, by theorems 1.1 and 2.2, we have

$$\sum_{v \in V} d(v) = 2\varepsilon = 2\nu - 2$$

It now follows that $d(v) = 1$ for at least two vertices v □

Exercises

1 Show that if any two vertices of a loopless graph G are connected by a unique path, then G is a tree.

2 Prove corollary 2.2 by showing that the origin and terminus of a longest path in a nontrivial tree both have degree one.

4 Show that every tree with exactly two vertices of degree one is a path.

5 Let G be a graph with $\nu - 1$ edges. Show that the following three statements are equivalent:

(*a*) G is connected;
(*b*) G is acyclic;
(*c*) G is a tree.

6 Show that if G is a tree with $\Delta \geq k$, then G has at least k vertices of degree one.

7 An acyclic graph is also called a *forest*. Show that

(*a*) each component of a forest is a tree;
(*b*) G is a forest if and only if $\varepsilon = \nu - \omega$.