Graph Theory

Loana Tito Nogueira

VERTEX DEGREES

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Proof Consider the incidence matrix **M**. The sum of the entries in the row corresponding to vertex v is precisely $d(v)$, and therefore $\sum_{v \in V} d(v)$ is just the sum of all entries in **M**. But this sum is also 2ε, since (exercise 1.3.1*a*) each of the ε column sums of **M** is 2 □

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A graph G is *k-regular* if $d(v) = k$ for all $v \in V$; a *regular graph* is one that is k-regular for some k. Complete graphs and complete bipartite graphs $K_{n,n}$ are regular; so, also, are the k-cubes.

Exercises

.1 Show that $\delta \leq 2\varepsilon/\nu \leq \Delta$.

.2 Show that if G is simple, the entries on the diagonals of both $\mathbf{MM}'$ and $\mathbf{A}^2$ are the degrees of the vertices of G.

.3 Show that if a k-regular bipartite graph with $k > 0$ has bipartition (X, Y), then $|X| = |Y|$.

.4 Show that, in any group of two or more people, there are always two with exactly the same number of friends inside the group.

5 If G has vertices $v_1, v_2, \ldots, v_\nu$, the sequence $(d(v_1), d(v_2), \ldots, d(v_\nu))$ is called a *degree sequence* of G. Show that a sequence $(d_1, d_2, \ldots, d_n)$ of non-negative integers is a degree sequence of some graph if and only if $\sum_{i=1}^{n} d_i$ is even.

$i=1$

.6 A sequence $\mathbf{d}=(d_1, d_2, \ldots, d_n)$ is *graphic* if there is a simple graph with degree sequence $\mathbf{d}$. Show that

(*a*) the sequences (7, 6, 5, 4, 3, 3, 2) and (6, 6, 5, 4, 3, 3, 1) are not graphic;

(*b*) if $\mathbf{d}$ is graphic and $d_1 \geq d_2 \geq \ldots \geq d_n$, then $\sum_{i=1}^{n} d_i$ is even and

$$\sum_{i=1}^{k} d_i \leq k(k-1) + \sum_{i=k+1}^{n} \min\{k, d_i\} \quad \text{for} \quad 1 \leq k \leq n$$

(Erdös and Gallai, 1960 have shown that this necessary condition is also sufficient for $\mathbf{d}$ to be graphic.)

7 Let $\mathbf{d}=(d_1, d_2, \ldots, d_n)$ be a nonincreasing sequence of non-negative integers, and denote the sequence $(d_2-1, d_3-1, \ldots, d_{d_1+1}-1, d_{d_1+2}, \ldots, d_n)$ by $\mathbf{d}'$.

(*a*)* Show that $\mathbf{d}$ is graphic if and only if $\mathbf{d}'$ is graphic.

(*b*) Using (*a*), describe an algorithm for constructing a simple graph with degree sequence $\mathbf{d}$, if such a graph exists.

(V. Havel, S. Hakimi)

8 The *edge graph* of a graph G is the graph with vertex set $E(G)$ in which two vertices are joined if and only if they are adjacent edges in G. Show that, if G is simple

(*a*) the edge graph of G has $\varepsilon(G)$ vertices and $\sum_{v \in V(G)} \binom{d_G(v)}{2}$ edges

PATHS AND CONNECTION

A *walk* in G is a finite non-null sequence $W = v_0e_1v_1e_2v_2 \ldots e_kv_k$, whose terms are alternately vertices and edges, such that, for $1 \leq i \leq k$, the ends of e_i are v_{i-1} and v_i. We say that W is a walk *from* v_0 *to* v_k, or a (v_0, v_k)-*walk*. The vertices v_0 and v_k are called the *origin* and *terminus* of W, respectively, and $v_1, v_2, \ldots, v_{k-1}$ its *internal vertices*. The integer k is the *length* of W.

If $W = v_0e_1v_1 \ldots e_kv_k$ and $W' = v_ke_{k+1}v_{k+1} \ldots e_lv_l$ are walks, the walk $v_ke_kv_{k-1} \ldots e_1v_0$, obtained by reversing W, is denoted by W^{-1} and the walk $v_0e_1v_1 \ldots e_lv_l$, obtained by concatenating W and W' at v_k, is denoted by WW'. A *section* of a walk $W = v_0e_1v_1 \ldots e_kv_k$ is a walk that is a subsequence $v_ie_{i+1}v_{i+1} \ldots e_jv_j$ of consecutive terms of W; we refer to this subsequence as the (v_i, v_j)-*section* of W.

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In a simple graph, a walk $v_0e_1v_1 \ldots e_kv_k$ is determined by the sequence $v_0v_1 \ldots v_k$ of its vertices; hence a walk in a simple graph can be specified simply by its vertex sequence. Moreover, even in graphs that are not simple, If the edges $e_1, e_2, \ldots, e_k$ of a walk W are distinct, W is called a *trail*; in this case the length of W is just $\varepsilon(W)$. If, in addition, the vertices $v_0, v_1, \ldots, v_k$ are distinct, W is called a *path*. Figure 1.8 illustrates a walk, a trail and a path in a graph. We shall also use the word 'path' to denote a graph or subgraph whose vertices and edges are the terms of a path.

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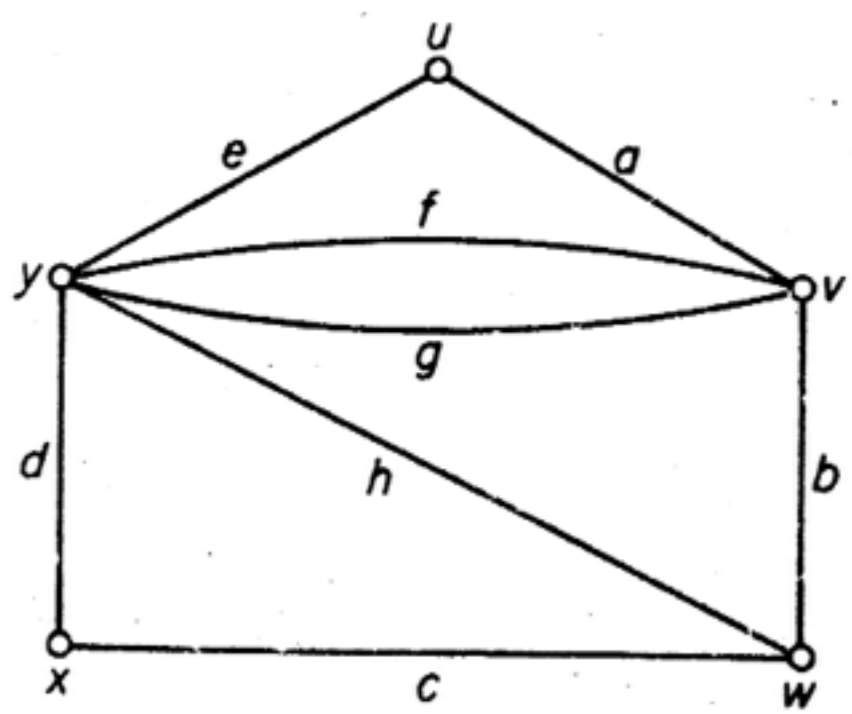


Walk: *uavfyfvgyhwbv*
Trail: *wcxdyhwbvgy*
Path: *xcwhyeuav*

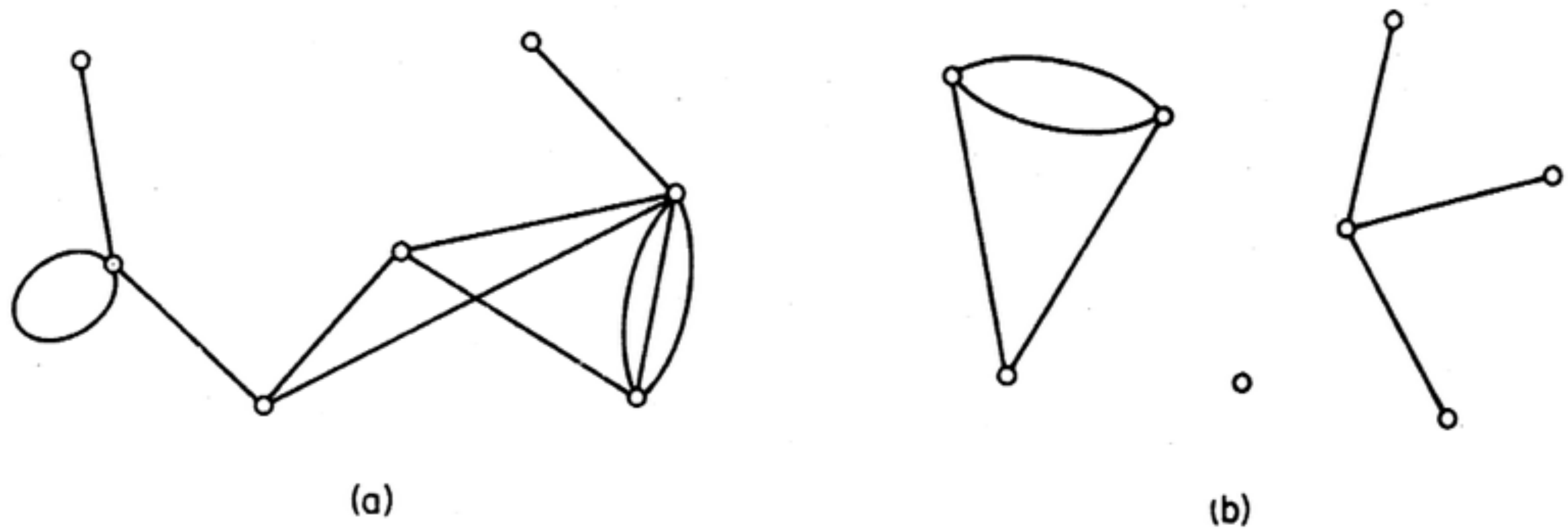
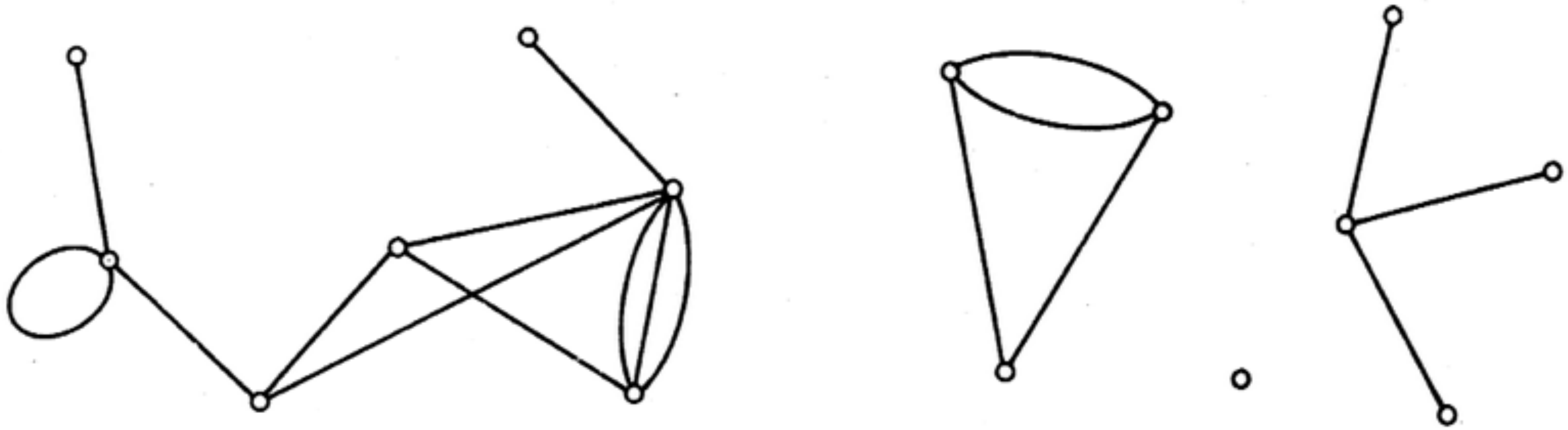


Figure 1.9. (*a*) A connected graph; (*b*) a disconnected graph with three components

Two vertices u and v of G are said to be *connected* if there is a (u, v)-path in G.



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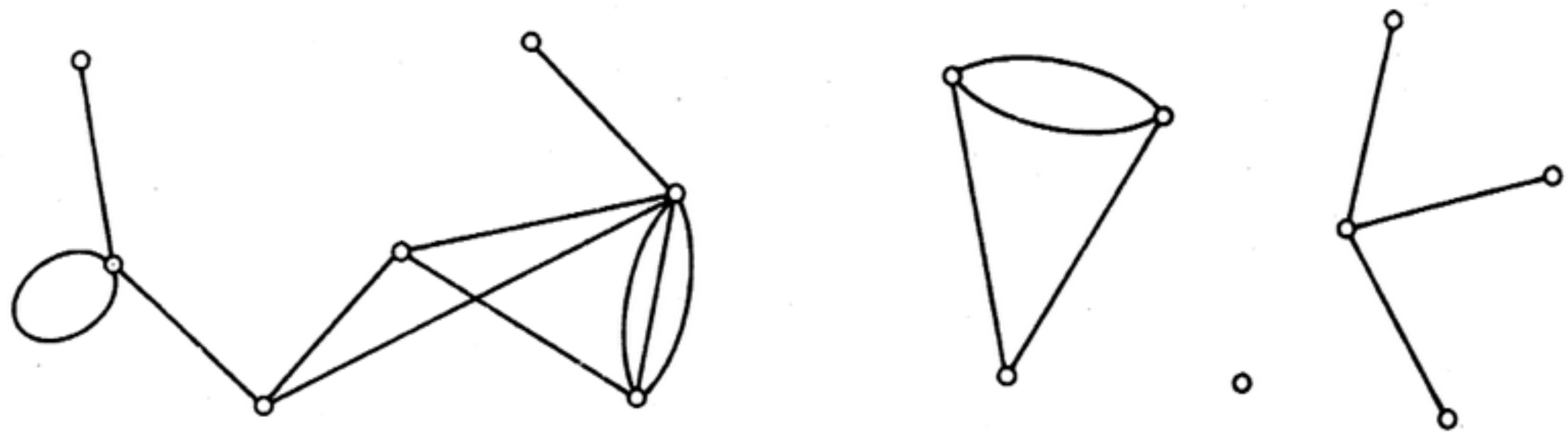
Two vertices u and v of G are said to be *connected* if there is a (u, v)-path in G. Connection is an equivalence relation on the vertex set V. Thus there is a partition of V into nonempty subsets $V_1, V_2, \ldots, V_\omega$ such that two vertices u and v are connected if and only if both u and v belong to the same set V_i.



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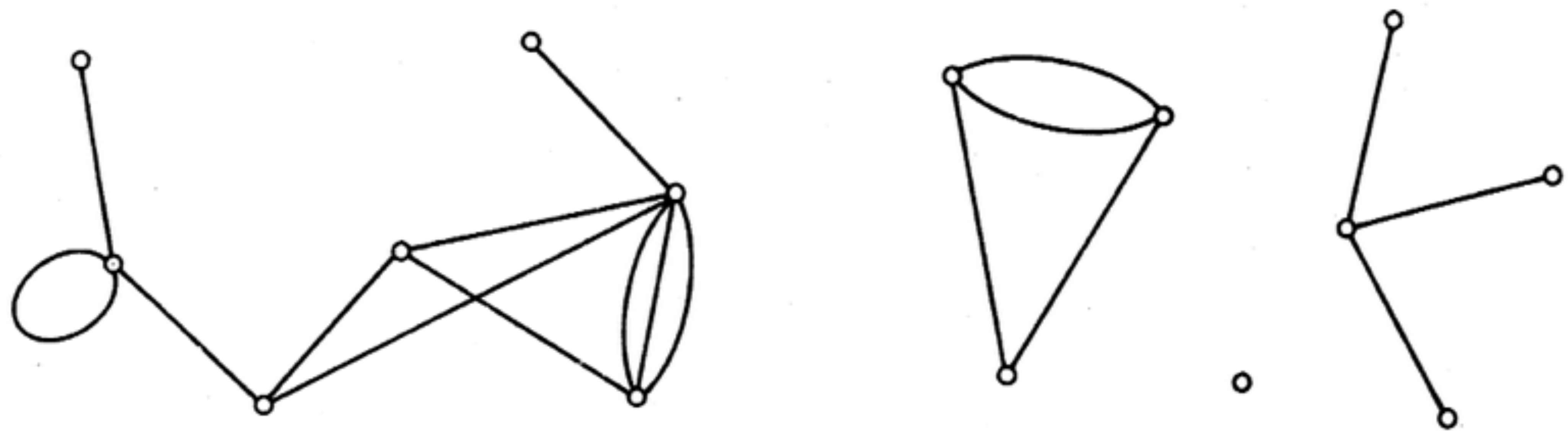

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1 Show that if there is a (u, v)-walk in G, then there is also a (u, v)-path in G.

2 Show that the number of (v_i, v_j)-walks of length k in G is the (i, j)th entry of $\mathbf{A}^k$.

3 Show that if G is simple and $\delta \geq k$, then G has a path of length k.

4 Show that G is connected if and only if, for every partition of V into two nonempty sets V_1 and V_2, there is an edge with one end in V_1 and one end in V_2.

5 (a) Show that if G is simple and $\varepsilon > \binom{\nu - 1}{2}$, then G is connected.

(b) For $\nu > 1$, find a disconnected simple graph G with $\varepsilon = \binom{\nu - 1}{2}$.

6 (a) Show that if G is simple and $\delta > [\nu/2] - 1$, then G is connected.

(b) Find a disconnected $([\nu/2] - 1)$-regular simple graph for ν even.

7 Show that if G is disconnected, then G^c is connected.

8 (a) Show that if $e \in E$, then $\omega(G) \leq \omega(G - e) \leq \omega(G) + 1$.

(b) Let $v \in V$. Show that $G - e$ cannot, in general, be replaced by $G - v$ in the above inequality.

9 Show that if G is connected and each degree in G is even, then, for any $v \in V$, $\omega(G - v) \leq \frac{1}{2} d(v)$.

10 Show that any two longest paths in a connected graph have a vertex in common.

11 If vertices u and v are connected in G, the *distance* between u and v in G, denoted by $d_G(u, v)$, is the length of a shortest (u, v)-path in G; if there is no path connecting u and v we define $d_G(u, v)$ to be infinite. Show that, for any three vertices u, v and w, $d(u, v) + d(v, w) \geq d(u, w)$.

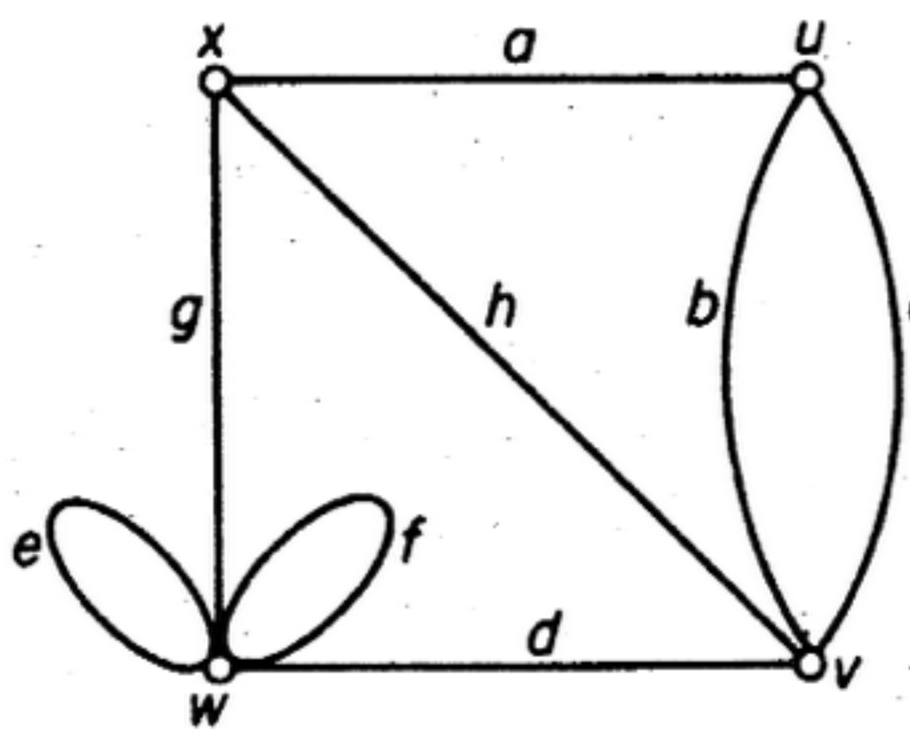
12 The *diameter* of G is the maximum distance between two vertices of G. Show that if G has diameter greater than three, then G^c has diameter less than three.

13 Show that if G is simple with diameter two and $\Delta = \nu - 2$, then $\varepsilon \geq 2\nu - 4$.

.14 Show that if G is simple and connected but not complete, then G has three vertices u, v and w such that $uv, vw \in E$ and $uw \notin E$.

CYCLES

A walk is *closed* if it has positive length and its origin and terminus are the same. A closed trail whose origin and internal vertices are distinct is a *cycle*. Just as with paths we sometimes use the term 'cycle' to denote a graph corresponding to a cycle. A cycle of length k is called a k-*cycle;* a k-cycle is *odd* or *even* according as k is odd or even. A 3-cycle is often called a *triangle*. Examples of a closed trail and a cycle are given in figure 1.10.



Closed trail: *ucvhxgwfwdvbu*

Cycle: *xaubvhx*

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Using the concept of a cycle, we can now present a characterisation of bipartite graphs.

Theorem 1.2 A graph is bipartite if and only if it contains no odd cycle.

Proof Suppose that G is bipartite with bipartition (X, Y), and let $C = v_0 v_1 \ldots v_k v_0$ be a cycle of G. Without loss of generality we may assume that $v_0 \in X$. Then, since $v_0 v_1 \in E$ and G is bipartite, $v_1 \in Y$. Similarly $v_2 \in X$ and, in general, $v_{2i} \in X$ and $v_{2i+1} \in Y$. Since $v_0 \in X$, $v_k \in Y$. Thus $k = 2i+1$, for some i, and it follows that C is even.

It clearly suffices to prove the converse for connected graphs. Let G be a connected graph that contains no odd cycles. We choose an arbitrary vertex u and define a partition (X, Y) of V by setting

$$X = \{x \in V \mid d(u, x) \quad \text{is even}\}$$
$$Y = \{y \in V \mid d(u, y) \quad \text{is odd}\}$$

We shall show that (X, Y) is a bipartition of G. Suppose that v and w are two vertices of X. Let P be a shortest (u, v)-path and Q be a shortest (u, w)-path. Denote by u_1 the last vertex common to P and Q. Since P and Q are shortest paths, the (u, u_1)-sections of both P and Q are shortest (u, u_1)-paths and, therefore, have the same length. Now, since the lengths of both P and Q are even, the lengths of the (u_1, v)-section P_1 of P and the (u_1, w)-section Q_1 of Q must have the same parity. It follows that the (v, w)-path $P_1^{-1}Q_1$ is of even length. If v were joined to w, $P_1^{-1}Q_1wv$ would be a cycle of odd length, contrary to the hypothesis. Therefore no two vertices in X are adjacent; similarly, no two vertices in Y are adjacent □