Graph Theory

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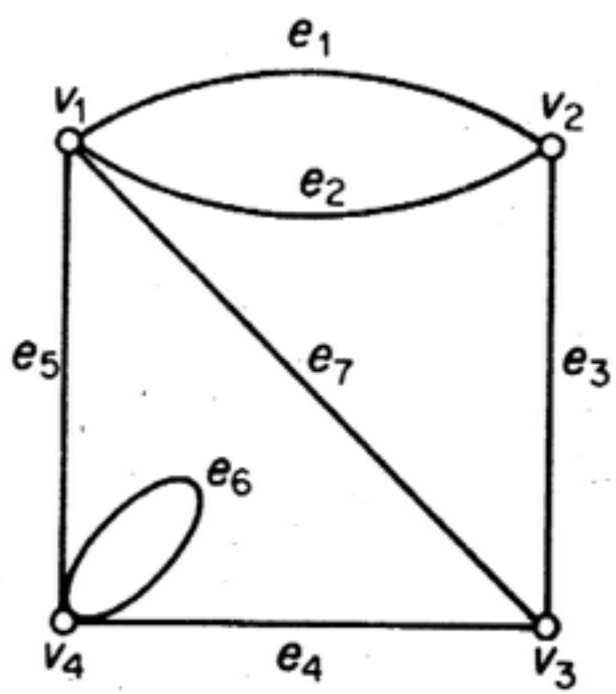
THE INCIDENCE AND ADJACENCY MATRICES

To any graph G there corresponds a $\nu \times \varepsilon$ matrix called the incidence matrix of G. Let us denote the vertices of G by $v_1, v_2, \ldots, v_\nu$ and the edges by $e_1, e_2, \ldots, e_\varepsilon$. Then the *incidence matrix* of G is the matrix $\mathbf{M}(G) = [m_{ij}]$, where m_{ij} is the number of times (0, 1 or 2) that v_i and e_j are incident. The incidence matrix of a graph is just a different way of specifying the graph.

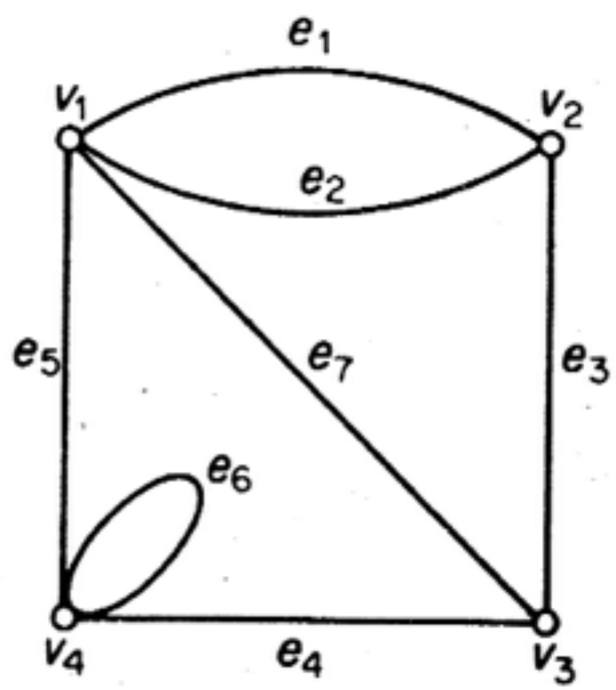


G

	e_1	e_2	e_3	e_4	e_5	e_6	e_7
v_1	1	1	0	0	1	0	1
v_2	1	1	1	0	0	0	0
v_3	0	0	1	1	0	0	1
v_4	0	0	0	1	1	2	0

$\mathbf{M}(G)$

Another matrix associated with G is the *adjacency matrix*; this is the $\nu \times \nu$ matrix $\mathbf{A}(G) = [a_{ij}]$, in which a_{ij} is the number of edges joining v_i and v_j. A



G

	v_1	v_2	v_3	v_4
v_1	0	2	1	1
v_2	2	0	1	0
v_3	1	1	0	1
v_4	1	0	1	1

$\mathbf{A}(G)$

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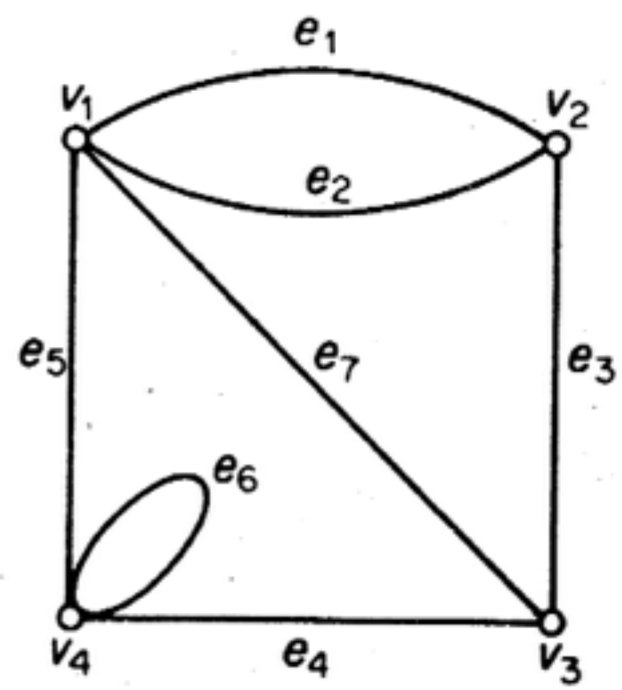

G

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v_3	1	1	0	1
v_4	1	0	1	1

A(G)

The adjacency matrix of a graph is generally considerably smaller than its incidence matrix, and it is in this form that graphs are commonly stored in computers.

Exercises

1 Let $\mathbf{M}$ be the incidence matrix and $\mathbf{A}$ the adjacency matrix of a graph G.

(*a*) Show that every column sum of $\mathbf{M}$ is 2.
(*b*) What are the column sums of $\mathbf{A}$?

2 Let G be bipartite. Show that the vertices of G can be enumerated so that the adjacency matrix of G has the form

$$\left[\begin{array}{c|c} \mathbf{0} & \mathbf{A}_{12} \\ \hline \mathbf{A}_{21} & \mathbf{0} \end{array}\right]$$

where $\mathbf{A}_{21}$ is the transpose of $\mathbf{A}_{12}$.

SUBGRAPHS

A graph H is a *subgraph* of G (written $H \subseteq G$) if $V(H) \subseteq V(G)$, $E(H) \subseteq E(G)$, and ψ_H is the restriction of ψ_G to $E(H)$. When $H \subseteq G$ but $H \neq G$, we write $H \subset G$ and call H a *proper subgraph* of G. If H is a subgraph of G, G is a *supergraph* of H. A *spanning subgraph* (or *spanning supergraph*) of G is a subgraph (or supergraph) H with $V(H) = V(G)$.

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By deleting from G all loops and, for every pair of adjacent vertices, all but one link joining them, we obtain a simple spanning subgraph of G, called the *underlying simple graph* of G. Figure 1.6 shows a graph and its underlying simple graph.

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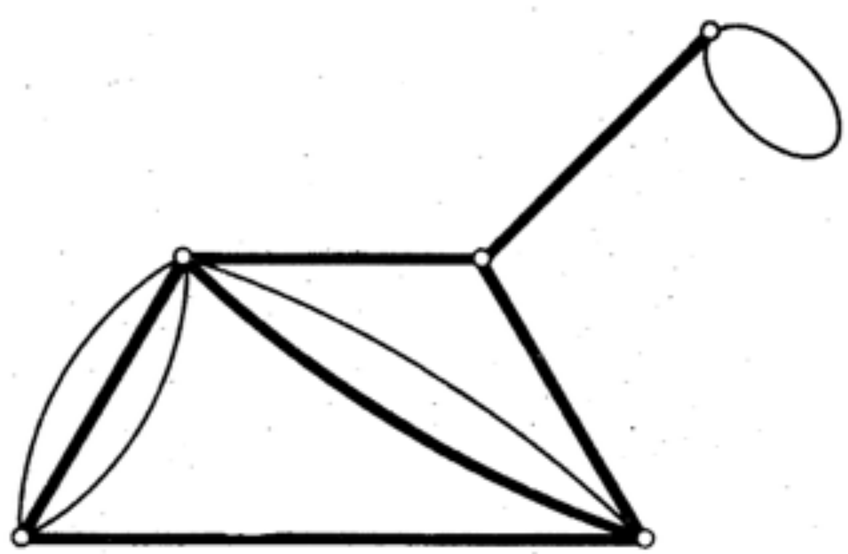
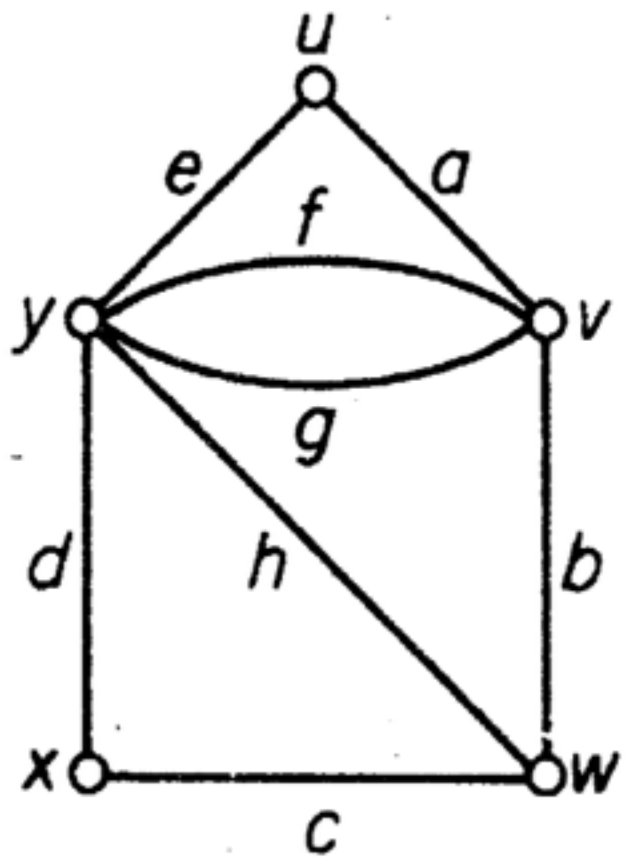
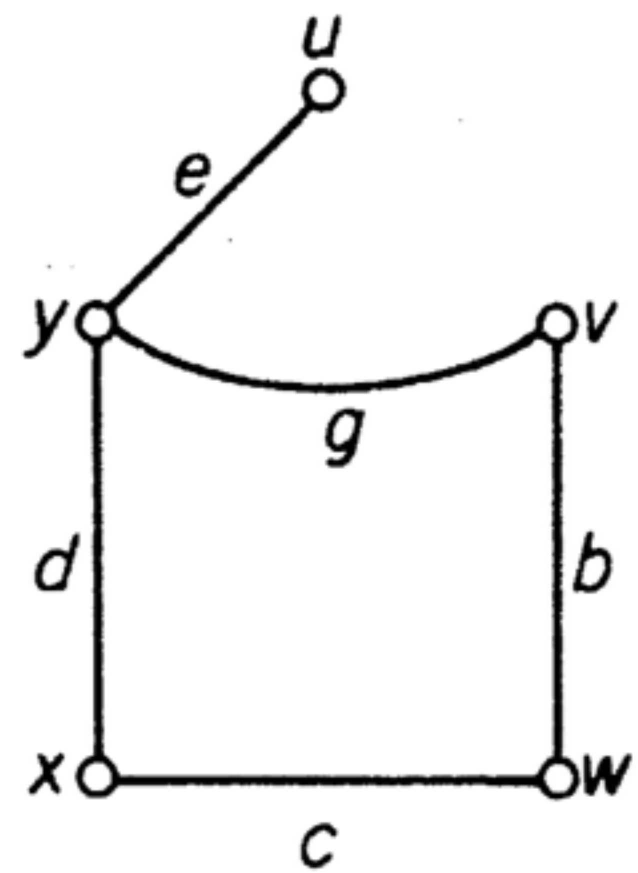


Graphs and Subgraphs



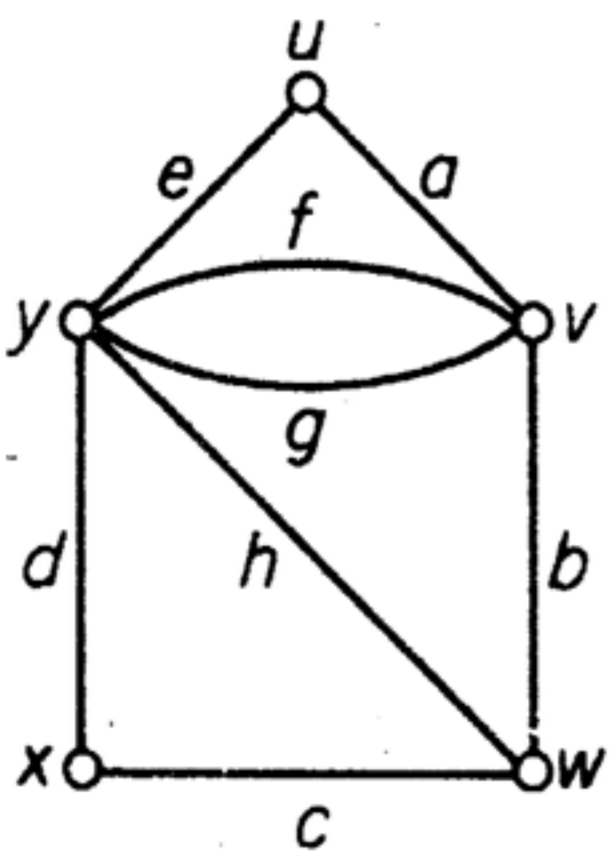
G



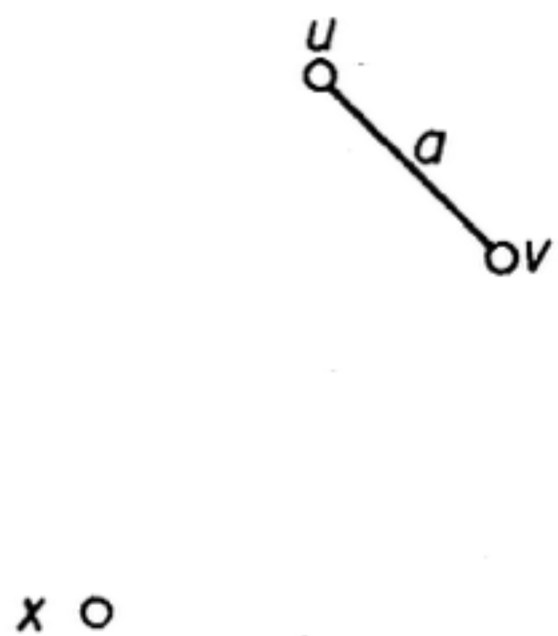
A spanning subgraph of G

SUBGRAPHS

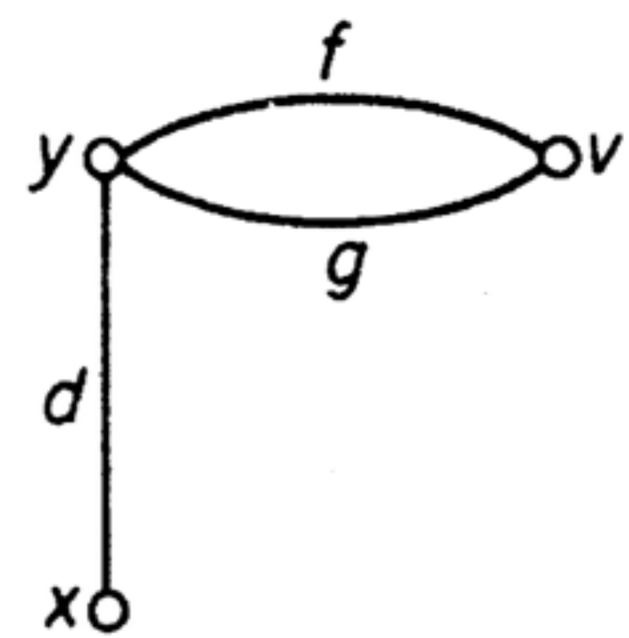
Suppose that V' is a nonempty subset of V. The subgraph of G whose vertex set is V' and whose edge set is the set of those edges of G that have both ends in V' is called the subgraph of G *induced* by V' and is denoted by $G[V']$; we say that $G[V']$ is an *induced subgraph* of G. The induced subgraph $G[V \backslash V']$ is denoted by $G - V'$; it is the subgraph obtained from G by deleting the vertices in V' together with their incident edges. If $V' = \{v\}$ we write $G - v$ for $G - \{v\}$.



G

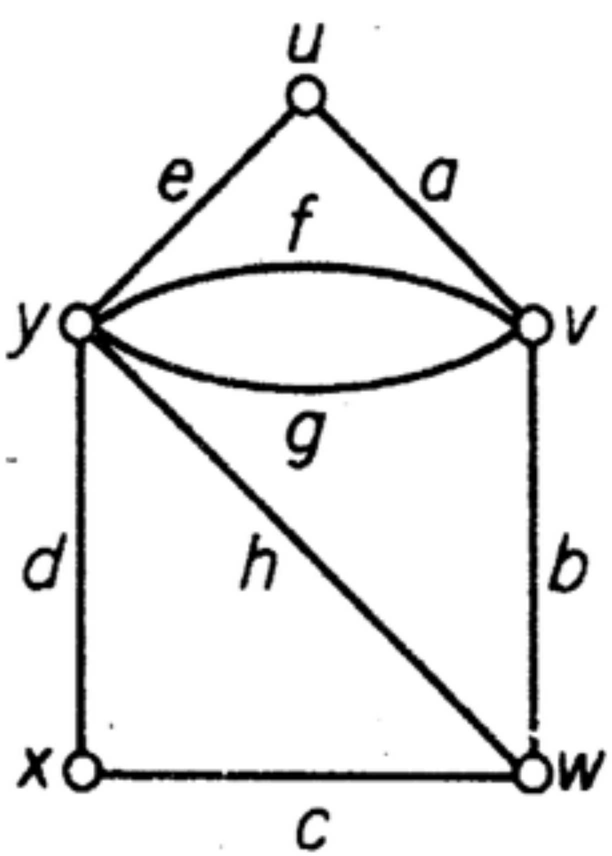


The induced subgraph $G[\{u, v, x\}]$

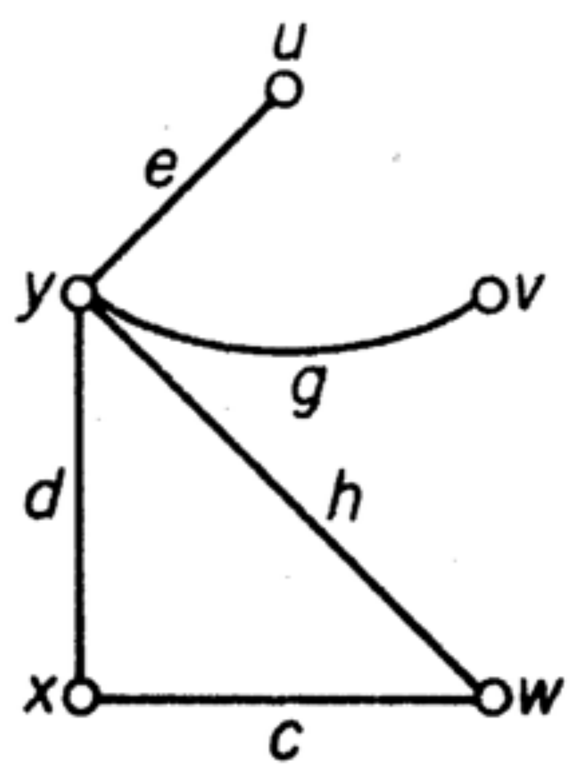


$G - \{u, w\}$

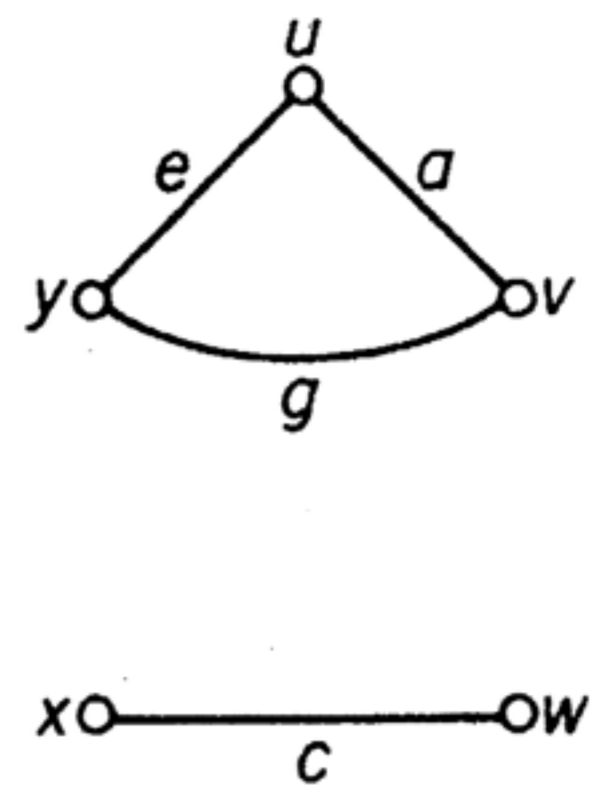
Now suppose that E' is a nonempty subset of E. The subgraph of G whose vertex set is the set of ends of edges in E' and whose edge set is E' is called the subgraph of G *induced* by E' and is denoted by $G[E']$; $G[E']$ is an *edge-induced subgraph* of G. The spanning subgraph of G with edge set $E \backslash E'$ is written simply as $G-E'$; it is the subgraph obtained from G by deleting the edges in E'. Similarly, the graph obtained from G by adding a set of edges E' is denoted by $G+E'$. If $E'=\{e\}$ we write $G-e$ and $G+e$ instead of $G-\{e\}$ and $G+\{e\}$.



G



$G-\{a, b, f\}$



The edge-induced subgraph $G[\{a, c, e, g\}]$

Let G_1 and G_2 be subgraphs of G. We say that G_1 and G_2 are *disjoint* if they have no vertex in common, and *edge-disjoint* if they have no edge in common. The *union* $G_1 \cup G_2$ of G_1 and G_2 is the subgraph with vertex set

$V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2)$; if G_1 and G_2 are disjoint, we sometimes denote their union by $G_1 + G_2$. The *intersection* $G_1 \cap G_2$ of G_1 and G_2 is defined similarly, but in this case G_1 and G_2 must have at least one vertex in common.

Exercises

1 Show that every simple graph on n vertices is isomorphic to a subgraph of K_n.

2 Show that

(*a*) every induced subgraph of a complete graph is complete;
(*b*) every subgraph of a bipartite graph is bipartite.

3 Describe how $\mathbf{M}(G-E')$ and $\mathbf{M}(G-V')$ can be obtained from $\mathbf{M}(G)$, and how $\mathbf{A}(G-V')$ can be obtained from $\mathbf{A}(G)$.

4 Find a bipartite graph that is not isomorphic to a subgraph of any k-cube.