Graph Theory

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Independent Sets and Cliques

A subset S of V is called an *independent set* of G if no two vertices of S are adjacent in G. An independent set is *maximum* if G has no independent set S' with $|S'|>|S|$. Examples of independent sets are shown in figure 7.1.

Recall that a subset K of V such that every edge of G has at least one end in K is called a covering of G. The two examples of independent sets given in figure 7.1 are both complements of coverings. It is not difficult to see that this is always the case.

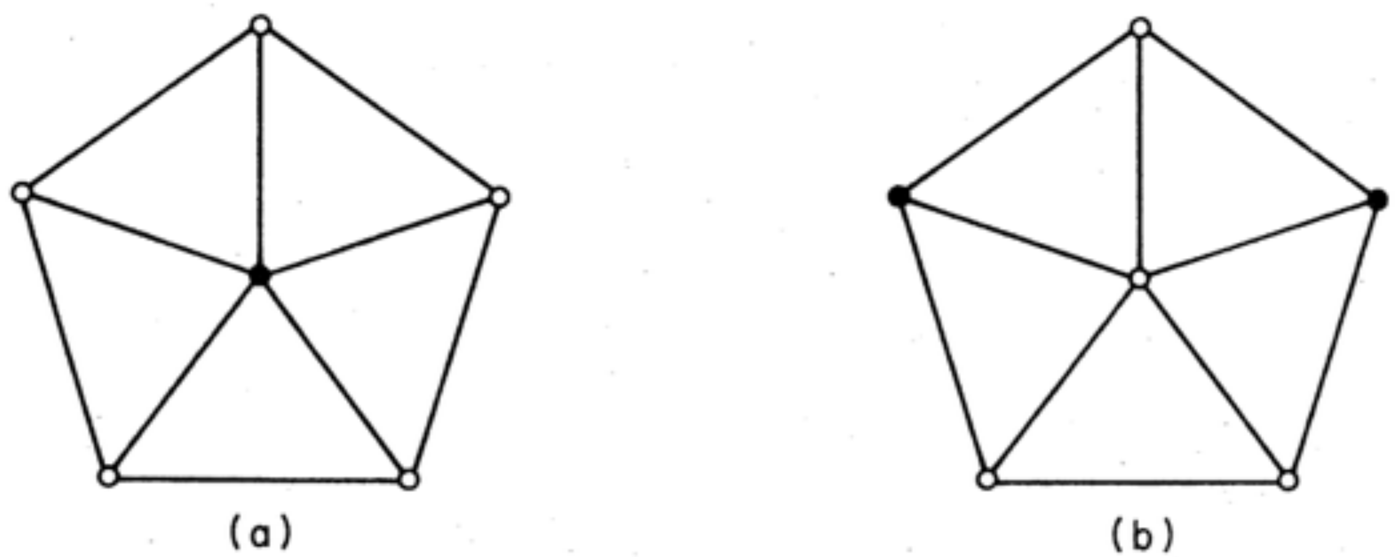

Figure 7.1. (a) An independent set; (b) a maximum independent set

Theorem 7.1 A set $S \subseteq V$ is an independent set of G if and only if $V \backslash S$ is a covering of G.

Proof By definition, S is an independent set of G if and only if no edge of G has both ends in S or, equivalently, if and only if each edge has at least one end in $V \backslash S$. But this is so if and only if $V \backslash S$ is a covering of G □

The number of vertices in a maximum independent set of G is called the *independence number* of G and is denoted by $\alpha(G)$; similarly, the number of vertices in a minimum covering of G is the *covering number* of G and is denoted by $\beta(G)$.

Corollary 7.1 $\alpha + \beta = \nu$.

Proof Let S be a maximum independent set of G, and let K be a minimum covering of G. Then, by theorem 7.1, $V \backslash K$ is an independent set and $V \backslash S$ is a covering. Therefore

$$\nu - \beta = |V \backslash K| \leq \alpha \tag{7.1}$$

and

$$\nu - \alpha = |V \backslash S| \geq \beta \tag{7.2}$$

Combining (7.1) and (7.2) we have $\alpha + \beta = \nu$ □

The edge analogue of an independent set is a set of links no two of which are adjacent, that is, a matching. The edge analogue of a covering is called an edge covering. An *edge covering* of G is a subset L of E such that each vertex of G is an end of some edge in L. Note that edge coverings do not always exist; a graph G has an edge covering if and only if $\delta > 0$. We denote the number of edges in a maximum matching of G by $\alpha'(G)$, and the number of edges in a minimum edge covering of G by $\beta'(G)$; the numbers $\alpha'(G)$ and $\beta'(G)$ are the *edge independence number* and *edge covering number* of G, respectively.

Theorem 7.2 (Gallai, 1959) If $\delta > 0$, then $\alpha' + \beta' = \nu$.

Proof Let M be a maximum matching in G and let U be the set of M-unsaturated vertices. Since $\delta > 0$ and M is maximum, there exists a set E' of $|U|$ edges, one incident with each vertex in U. Clearly, $M \cup E'$ is an edge covering of G, and so

$$\beta' \leq |M \cup E'| = \alpha' + (\nu - 2\alpha') = \nu - \alpha'$$

or

$$\alpha' + \beta' \leq \nu \tag{7.3}$$

Now let L be a minimum edge covering of G, set $H = G[L]$ and let M be a maximum matching in H. Denote the set of M-unsaturated vertices in H by U. Since M is maximum, $H[U]$ has no links and therefore

$$|L| - |M| = |L \backslash M| \geq |U| = \nu - 2\,|M|$$

Because H is a subgraph of G, M is a matching in G and so

$$\alpha' + \beta' \geq |M| + |L| \geq \nu \tag{7.4}$$

Combining (7.3) and (7.4), we have $\alpha' + \beta' = \nu$ □

Theorem 7.3 In a bipartite graph G with $\delta > 0$, the number of vertices in a maximum independent set is equal to the number of edges in a minimum edge covering.

Proof Let G be a bipartite graph with $\delta > 0$. By corollary 7.1 and theorem 7.2, we have

$$\alpha + \beta = \alpha' + \beta'$$

and, since G is bipartite, it follows from theorem 5.3 that $\alpha' = \beta$. Thus $\alpha = \beta'$ $\square$

Even though the concept of an independent set is analogous to that of a matching, there exists no theory of independent sets comparable to the theory of matchings presented in chapter 5; for example, no good algorithm for finding a maximum independent set in a graph is known.