Graph Theory

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Matchings

A subset M of E is called a *matching* in G if its elements are links and no two are adjacent in G; the two ends of an edge in M are said to be *matched under* M. A matching M *saturates* a vertex v, and v is said to be M-*saturated*, if some edge of M is incident with v; otherwise, v is M-*unsaturated*. If every vertex of G is M-saturated, the matching M is *perfect*. M is a *maximum matching* if G has no matching M' with $|M'|>|M|$; clearly, every perfect matching is maximum. Maximum and perfect matchings in graphs are indicated in figure 5.1.

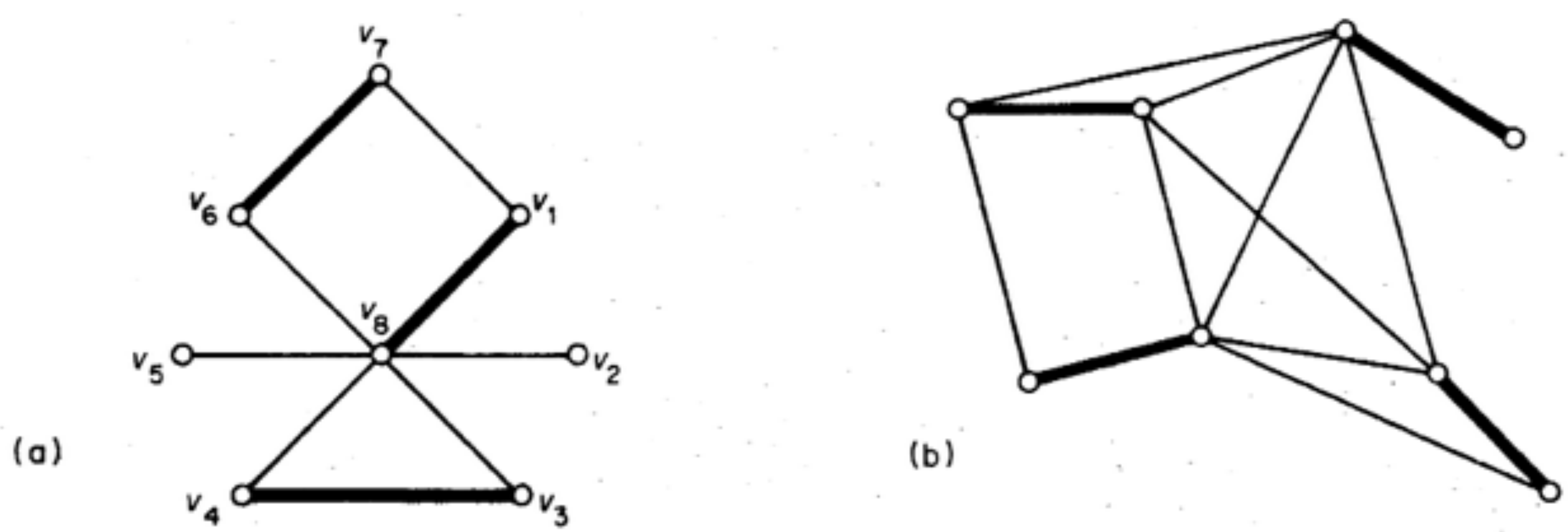
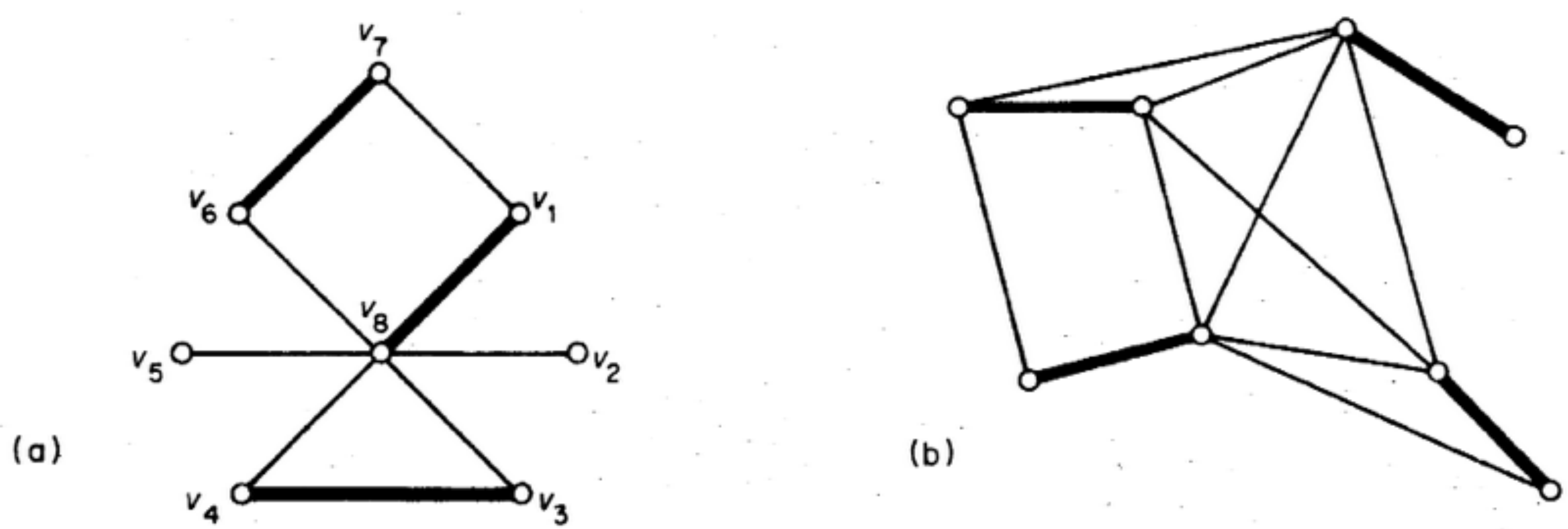


Figure 5.1. (*a*) A maximum matching; (*b*) a perfect matching

Let M be a matching in G. An *M-alternating path* in G is a path whose edges are alternately in $E\backslash M$ and M. For example, the path $v_5v_8v_1v_7v_6$ in the graph of figure 5.1a is an M-alternating path. An *M-augmenting path* is an M-alternating path whose origin and terminus are M-unsaturated.



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Proof Let M be a matching in G, and suppose that G contains an M-augmenting path $v_0v_1 \ldots v_{2m+1}$. Define $M' \subseteq E$ by

$$M' = (M \backslash \{v_1v_2, v_3v_4, \ldots, v_{2m-1}v_{2m}\}) \cup \{v_0v_1, v_2v_3, \ldots, v_{2m}v_{2m+1}\}$$

Then M' is a matching in G, and $|M'| = |M| + 1$. Thus M is not a maximum matching.

By Contrapositive

Conversely, suppose that M is not a maximum matching, and let M' be a maximum matching in G. Then

$$|M'| > |M| \tag{5.1}$$

Set $H = G[M \Delta M']$, where $M \Delta M'$ denotes the symmetric difference of M and M' (see figure 5.2).

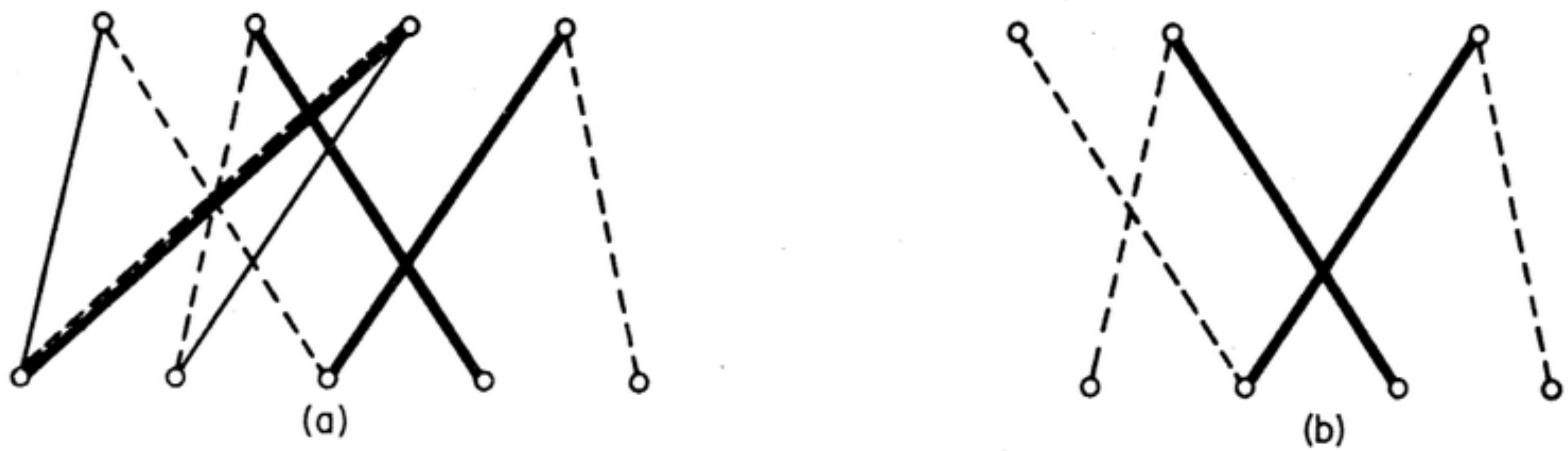

Figure 5.2. (*a*) G, with M heavy and M' broken; (*b*) $G[M \Delta M']$

Each vertex of H has degree either one or two in H, since it can be incident with at most one edge of M and one edge of M'. Thus each component of H is either an even cycle with edges alternately in M and M', or else a path with edges alternately in M and M'. By (5.1), H contains more edges of M' than of M, and therefore some path component P of H must start and end with edges of M'. The origin and terminus of P, being M'-saturated in H, are M-unsaturated in G. Thus P is an M-augmenting path in G □

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Suppose, now, that G is a bipartite graph with bipartition (X, Y). In many applications one wishes to find a matching of G that saturates every vertex in X; an example is the personnel assignment problem.

Theorem 5.2 Let G be a bipartite graph with bipartition (X, Y). Then G contains a matching that saturates every vertex in X if and only if

$$|N(S)| \geq |S| \quad \text{for all} \quad S \subseteq X \tag{5.2}$$

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Proof Suppose that G contains a matching M which saturates every vertex in X, and let S be a subset of X. Since the vertices in S are matched under M with distinct vertices in $N(S)$, we clearly have $|N(S)| \geq |S|$.

Conversely, suppose that G is a bipartite graph satisfying (5.2), but that G contains no matching saturating all the vertices in X. We shall obtain a contradiction. Let M^* be a maximum matching in G. By our supposition, M^* does not saturate all vertices in X. Let u be an M^*-unsaturated vertex in X, and let Z denote the set of all vertices connected to u by M^*-alternating paths. Since M^* is a maximum matching, it follows from theorem 5.1 that u is the only M^*-unsaturated vertex in Z. Set $S = Z \cap X$ and $T = Z \cap Y$ (see figure 5.3).

Clearly, the vertices in $S \backslash \{u\}$ are matched under M^* with the vertices in T. Therefore

$$|T| = |S| - 1 \tag{5.3}$$

and $N(S) \supseteq T$. In fact, we have

$$N(S) = T \tag{5.4}$$

since every vertex in $N(S)$ is connected to u by an M^*-alternating path. But

(5.3) and (5.4) imply that

$$|N(S)| = |S| - 1 < |S|$$

contradicting assumption (5.2) □

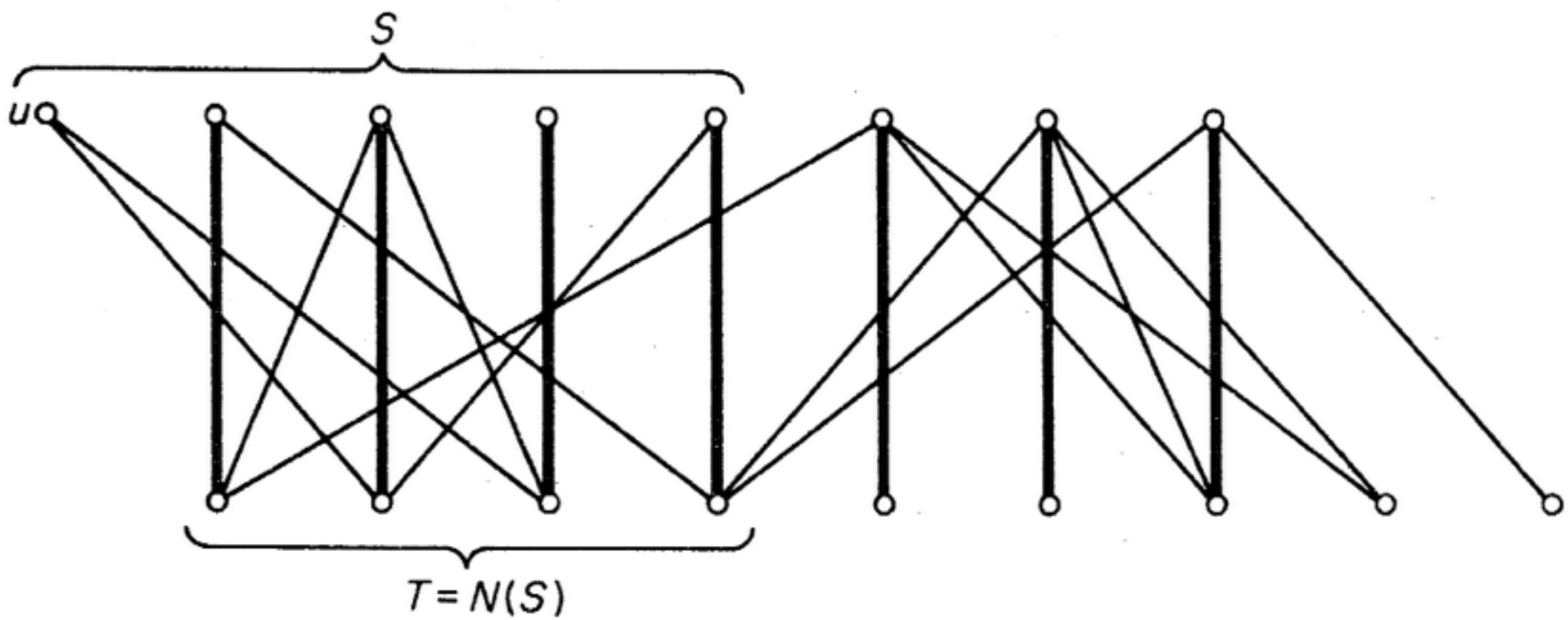

Figure 5.3

Corollary 5.2 If G is a k-regular bipartite graph with $k > 0$, then G has a perfect matching.

Proof Let G be a k-regular bipartite graph with bipartition (X, Y). Since G is k-regular, $k\,|X| = |E| = k\,|Y|$ and so, since $k > 0$, $|X| = |Y|$. Now let S be a subset of X and denote by E_1 and E_2 the sets of edges incident with vertices in S and $N(S)$, respectively. By definition of $N(S)$, $E_1 \subseteq E_2$ and therefore

$$k\,|N(S)| = |E_2| \geq |E_1| = k\,|S|$$

It follows that $|N(S)| \geq |S|$ and hence, by theorem 5.2, that G has a matching M saturating every vertex in X. Since $|X| = |Y|$, M is a perfect matching □

Corollary 5.2 is sometimes known as the *marriage theorem*, since it can be more colourfully restated as follows: if every girl in a village knows exactly k boys, and every boy knows exactly k girls, then each girl can marry a boy she knows, and each boy can marry a girl he knows.