Graph Theory

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Theorem 4.3 If G is a simple graph with $\nu \geq 3$ and $\delta \geq \nu/2$, then G is hamiltonian.

Proof By contradiction. Suppose that the theorem is false, and let G be a maximal nonhamiltonian simple graph with $\nu \geq 3$ and $\delta \geq \nu/2$. Since $\nu \geq 3$, G cannot be complete. Let u and v be nonadjacent vertices in G. By the choice of G, $G+uv$ is hamiltonian. Moreover, since G is nonhamiltonian, each Hamilton cycle of $G+uv$ must contain the edge uv. Thus there is a Hamilton path $v_1 v_2 \ldots v_\nu$ in G with origin $u = v_1$ and terminus $v = v_\nu$. Set

$$S = \{v_i \mid uv_{i+1} \in E\} \quad \text{and} \quad T = \{v_i \mid v_i v \in E\}$$

Since $v_\nu \notin S \cup T$ we have

$$|S \cup T| < \nu \tag{4.2}$$

Furthermore

$$|S \cap T| = 0 \tag{4.3}$$

since if $S \cap T$ contained some vertex v_i, then G would have the Hamilton cycle $v_1 v_2 \ldots v_i v_\nu v_{\nu-1} \ldots v_{i+1} v_1$, contrary to assumption (see figure 4.5).

Using (4.2) and (4.3) we obtain

$$d(u) + d(v) = |S| + |T| = |S \cup T| + |S \cap T| < \nu \tag{4.4}$$

But this contradicts the hypothesis that $\delta \geq \nu/2$ □

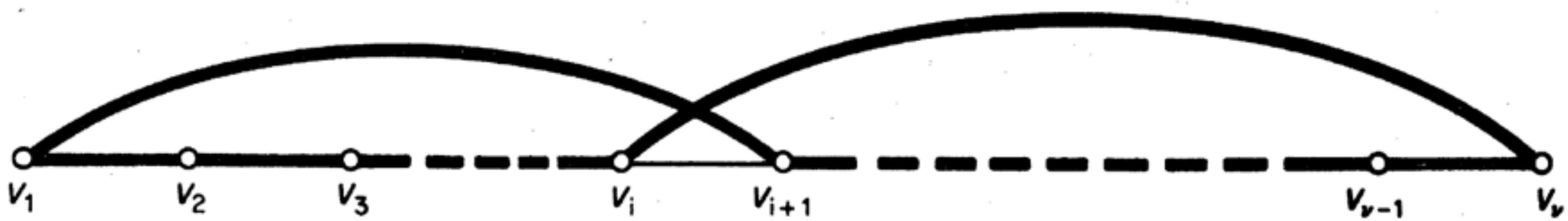

Figure 4.5

Bondy and Chvátal (1974) observed that the proof of theorem 4.3 can be modified to yield stronger sufficient conditions than that obtained by Dirac. The basis of their approach is the following lemma.

Lemma 4.4.1 Let G be a simple graph and let u and v be nonadjacent vertices in G such that

$$d(u)+d(v)\geq \nu \tag{4.5}$$

Then G is hamiltonian if and only if $G+uv$ is hamiltonian.

Proof If G is hamiltonian then, trivially, so too is $G+uv$. Conversely, suppose that $G+uv$ is hamiltonian but G is not. Then, as in the proof of theorem 4.3, we obtain (4.4). But this contradicts hypothesis (4.5) □

Lemma 4.4.1 motivates the following definition. The *closure* of G is the graph obtained from G by recursively joining pairs of nonadjacent vertices whose degree sum is at least ν until no such pair remains. We denote the closure of G by $c(G)$.

Lemma 4.4.2 $c(G)$ is well defined.

Proof Let G_1 and G_2 be two graphs obtained from G by recursively joining pairs of nonadjacent vertices whose degree sum is at least ν until no such pair remains. Denote by $e_1, e_2, \ldots, e_m$ and $f_1, f_2, \ldots, f_n$ the sequences of edges added to G in obtaining G_1 and G_2, respectively. We shall show that each e_i is an edge of G_2 and each f_j is an edge of G_1.

If possible, let $e_{k+1} = uv$ be the first edge in the sequence $e_1, e_2, \ldots, e_n$ that is not an edge of G_2. Set $H = G + \{e_1, e_2, \ldots, e_k\}$. It follows from the definition of G_1 that

$$d_H(u) + d_H(v) \geq \nu$$

By the choice of e_{k+1}, H is a subgraph of G_2. Therefore

$$d_{G_2}(u) + d_{G_2}(v) \geq \nu$$

This is a contradiction, since u and v are nonadjacent in G_2. Therefore each e_i is an edge of G_2 and, similarly, each f_j is an edge of G_1. Hence $G_1 = G_2$, and $c(G)$ is well defined □

Figure 4.6 illustrates the construction of the closure of a graph G on six vertices. It so happens that in this example $c(G)$ is complete; note, however, that this is by no means always the case.

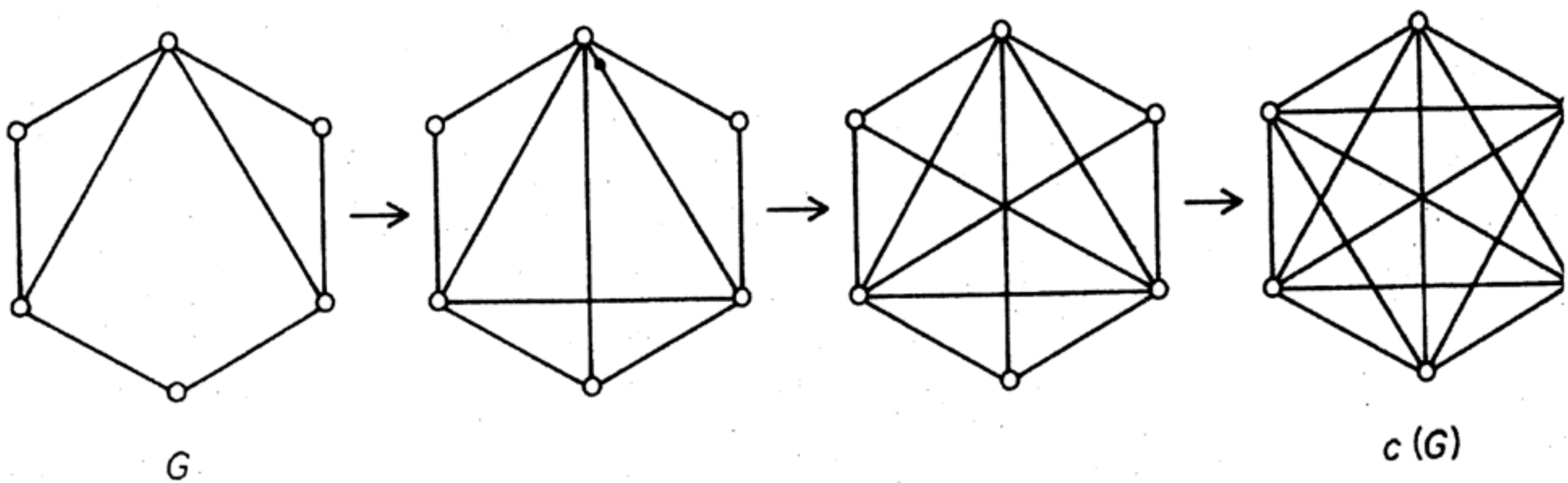

Figure 4.6. The closure of a graph

Theorem 4.4 A simple graph is hamiltonian if and only if its closure is hamiltonian.

Proof Apply lemma 4.4.1 each time an edge is added in the formation of the closure □

Theorem 4.4 has a number of interesting consequences. First, upon making the trivial observation that all complete graphs on at least three vertices are hamiltonian, we obtain the following result.

Corollary 4.4 Let G be a simple graph with $\nu \geq 3$. If $c(G)$ is complete, then G is hamiltonian.