Graph Theory

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Euler Tours and Hamilton Cycles

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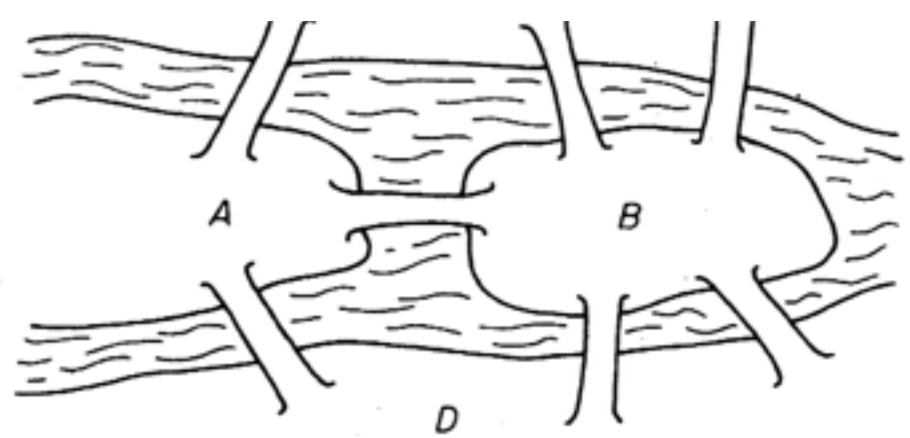

(a)

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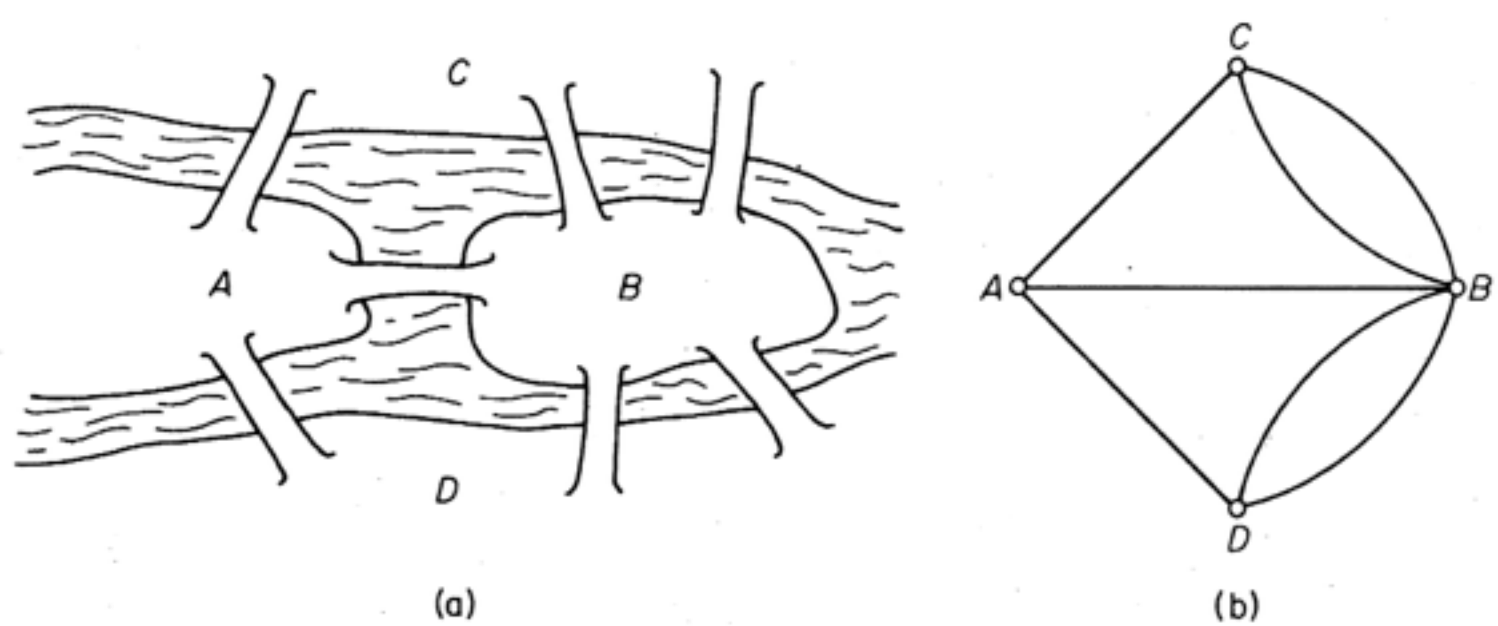

Figure 4.1. The bridges of Königsberg and their graph

A *tour* of G is a closed walk that traverses each edge of G at least once. An *Euler tour* is a tour which traverses each edge exactly once (in other words, a closed Euler trail). A graph is *eulerian* if it contains an Euler tour.

Theorem 4.1 A nonempty connected graph is eulerian if and only if it has no vertices of odd degree.

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Proof Let G be eulerian, and let C be an Euler tour of G with origin (and terminus) u. Each time a vertex v occurs as an internal vertex of C, two of the edges incident with v are accounted for. Since an Euler tour contains every edge of G, $d(v)$ is even for all $v \neq u$. Similarly, since C starts and ends at u, $d(u)$ is also even. Thus G has no vertices of odd degree.

Conversely, suppose that G is a noneulerian connected graph with at least one edge and no vertices of odd degree. Choose such a graph G with as few edges as possible. Since each vertex of G has degree at least two, G contains a closed trail (exercise 1.7.2). Let C be a closed trail of maximum possible length in G. By assumption, C is not an Euler tour of G and so $G-E(C)$ has some component G' with $\varepsilon(G')>0$. Since C is itself eulerian, it has no vertices of odd degree; thus the connected graph G' also has no vertices of odd degree. Since $\varepsilon(G')<\varepsilon(G)$, it follows from the choice of G that G' has an Euler tour C'. Now, because G is connected, there is a vertex v in $V(C)\cap V(C')$, and we may assume, without loss of generality, that v is the origin and terminus of both C and C'. But then CC' is a closed trail of G with $\varepsilon(CC')>\varepsilon(C)$, contradicting the choice of C □

Corollary 4.1 A connected graph has an Euler trail if and only if it has at most two vertices of odd degree.

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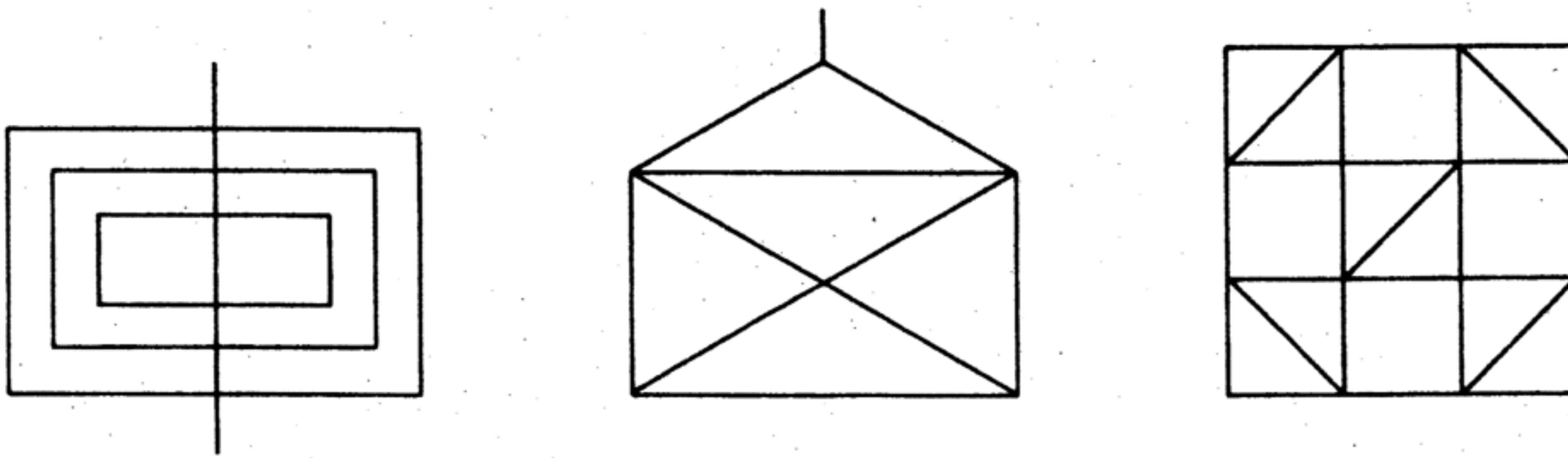
Proof If G has an Euler trail then, as in the proof of theorem 4.1, each vertex other than the origin and terminus of this trail has even degree.

Conversely, suppose that G is a nontrivial connected graph with at most two vertices of odd degree. If G has no such vertices then, by theorem 4.1, G has a closed Euler trail. Otherwise, G has exactly two vertices, u and v, of odd degree. In this case, let $G+e$ denote the graph obtained from G by the addition of a new edge e joining u and v. Clearly, each vertex of $G+e$ has even degree and so, by theorem 4.1, $G+e$ has an Euler tour $C = v_0e_1v_1 \ldots e_{\varepsilon+1}v_{\varepsilon+1}$, where $e_1 = e$. The trail $v_1e_2v_2 \ldots e_{\varepsilon+1}v_{\varepsilon+1}$ is an Euler trail of G □

Exercises

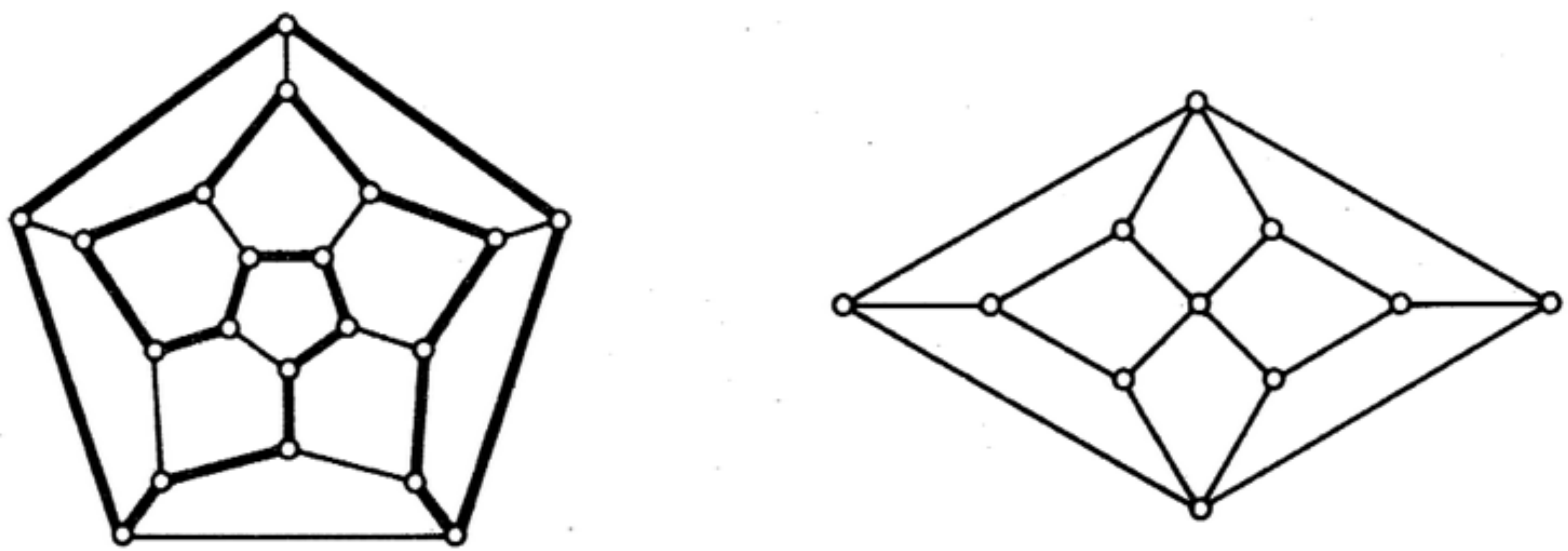
4.1.1 Which of the following figures can be drawn without lifting one's pen from the paper or covering a line more than once?



4.1.2 If possible, draw an eulerian graph G with ν even and ε odd; otherwise, explain why there is no such graph.

HAMILTON CYCLES

A path that contains every vertex of G is called a *Hamilton path* of G; similarly, a *Hamilton cycle* of G is a cycle that contains every vertex of G. Such paths and cycles are named after Hamilton (1856), who described, in a letter to his friend Graves, a mathematical game on the dodecahedron (figure 4.2*a*) in which one person sticks five pins in any five consecutive vertices and the other is required to complete the path so formed to a



(a) (b)

Figure 4.2. (*a*) The dodecahedron; (*b*) the Herschel graph

spanning cycle. A graph is *hamiltonian* if it contains a Hamilton cycle. The dodecahedron is hamiltonian (see figure 4.2*a*); the Herschel graph (figure 4.2*b*) is nonhamiltonian, because it is bipartite and has an odd number of vertices.

In contrast with the case of eulerian graphs, no nontrivial necessary and sufficient condition for a graph to be hamiltonian is known; in fact, the problem of finding such a condition is one of the main unsolved problems of graph theory.

We shall first present a simple, but useful, necessary condition.

Theorem 4.2 If G is hamiltonian then, for every nonempty proper subset S of V

$$\omega(G-S) \leq |S| \tag{4.1}$$

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Proof Let C be a Hamilton cycle of G. Then, for every nonempty proper subset S of V

$$\omega(C-S) \leq |S|$$

Also, $C-S$ is a spanning subgraph of $G-S$ and so

$$\omega(G-S) \leq \omega(C-S)$$

The theorem follows □

As an illustration of the above theorem, consider the graph of figure 4.3. This graph has nine vertices; on deleting the three indicated in black, four components remain. Therefore (4.1) is not satisfied and it follows from theorem 4.2 that the graph is nonhamiltonian.

We thus see that theorem 4.2 can sometimes be applied to show that a particular graph is nonhamiltonian. However, this method does not always

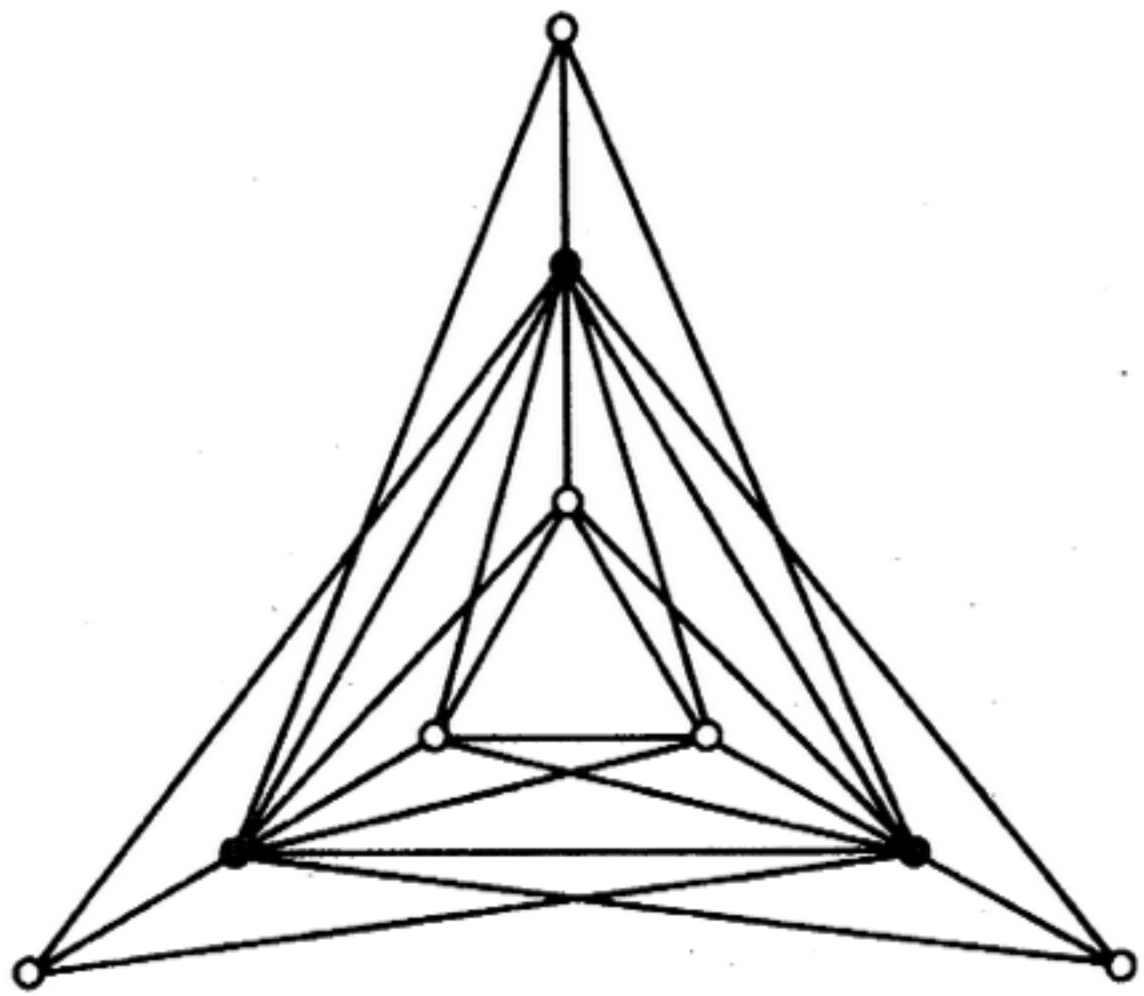

Figure 4.3

work; for instance, the Petersen graph (figure 4.4) is nonhamiltonian, but one cannot deduce this by using theorem 4.2.

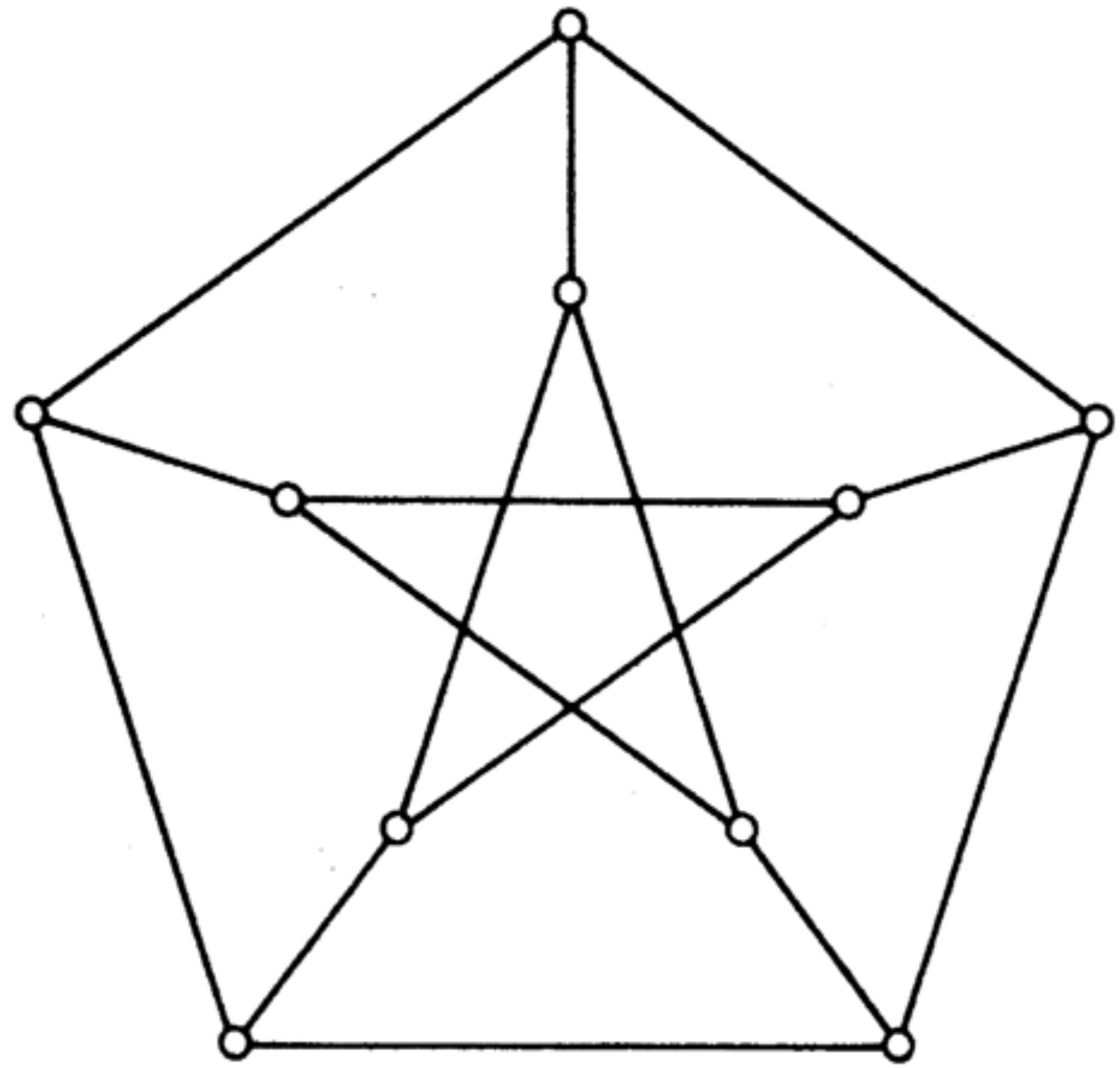

Figure 4.4. The Petersen graph