Graph Theory

Loana T. Nogueira

How this course is going to work?

Questions and answers in English

Exam will be in English

Presentation in English

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Everything in ENGLISH

What we are going to see in this course? When?

- Graphs and Subgraphs - 24/03
- Trees - 31/03
- Connectivity - 07/04
- 14/04 and 21/04 - Holiday
- Euler Tours and Hamiltonian cycles - 28/04
- Matchings - 05/05
- 12/05 - 1st Exam
- Edge colourings - 19/05
- Independent sets and cliques - 26/05
- Vertex Colouring - 02/06
- 09/06 or 23/06 - 2nd Exam
- 30/06 - Results and exam review

Main Bibliography

GRAPH THEORY WITH APPLICATIONS

J. A. Bondy and U. S. R. Murty

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Ontario, Canada

NORTH-HOLLAND
New York • Amsterdam • Oxford

What we are going to see in this course?

- **Graphs and Subgraphs**
    - Graphs and simple graphs
    - Graph isomorphism
    - The incidence and adjacency matrices
    - Subgraphs
    - Vertex Degrees
    - Paths

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Graphs and simple graphs

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Examples

1. Computer Networks

$$V = \{\text{computers}\}$$
$$E = \{(A, B) \mid \text{ computers } A \text{ and } B \text{ are networked}\}$$

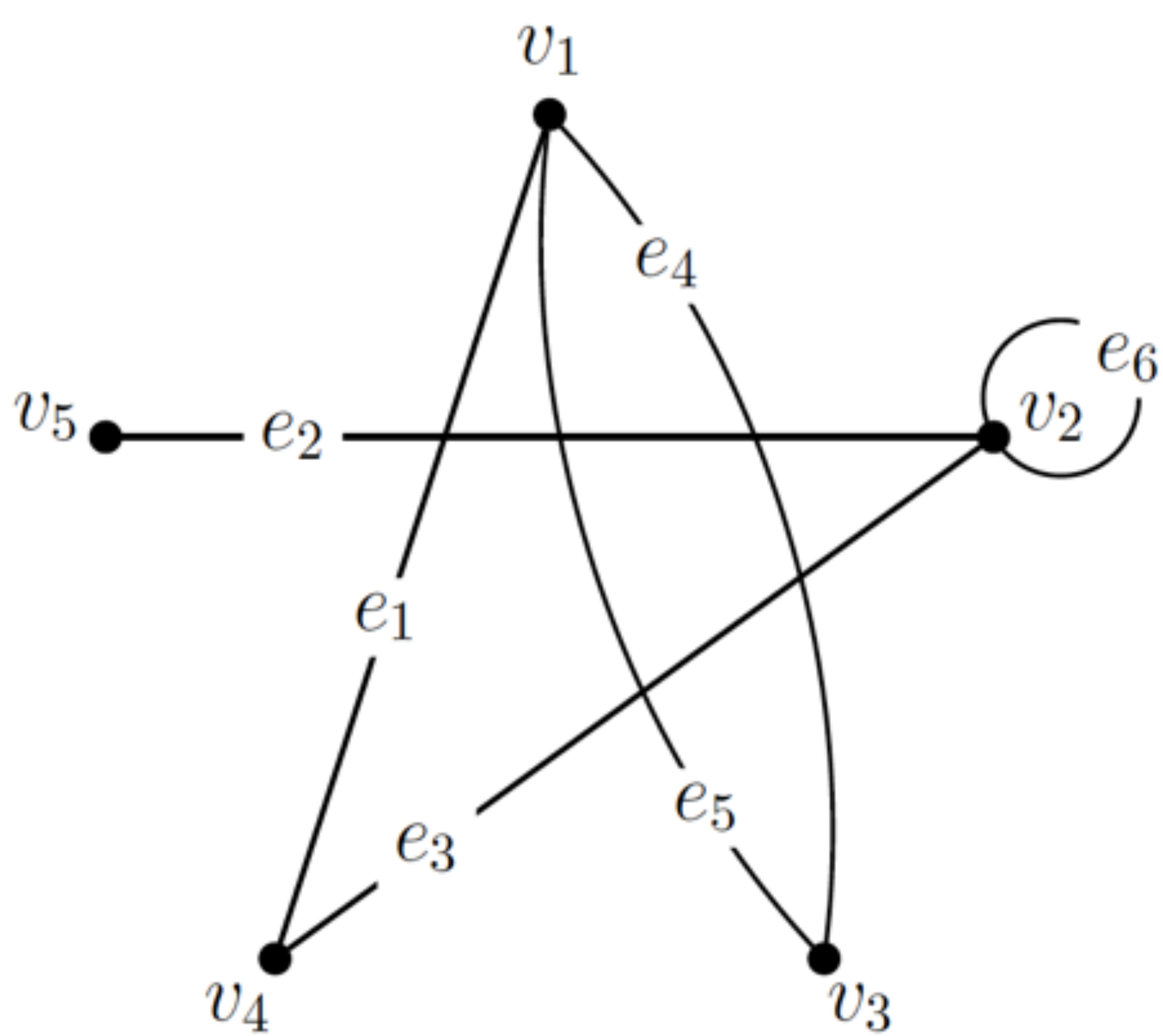
Graphs and simple graphs

2. Social Networks

$$
\begin{aligned}
V &= \{\text{Alice, Bob, Chris, Daniel, ...}\} \\
E &= \{(A, B) \mid A \text{ and } B \text{ know each other}\}
\end{aligned}
$$

Graphs and simple graphs

3. $V = \{v_1, v_2, v_3, v_4, v_5\}$ and $E = \{e_1, e_2, e_3, e_4, e_5, e_6\}$.



Graphs and simple graphs

Alternatively…

A *graph* G is an ordered triple $(V(G), E(G), \psi_G)$ consisting of a nonempty set $V(G)$ of *vertices*, a set $E(G)$, disjoint from $V(G)$, of *edges*, and an *incidence function* ψ_G that associates with each edge of G an unordered pair of (not necessarily distinct) vertices of G.

Graphs and simple graphs

Example 1

$$G = (V(G), E(G), \psi_G)$$

where

$$V(G) = \{v_1, v_2, v_3, v_4, v_5\}$$

$$E(G) = \{e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8\}$$

and ψ_G is defined by

$$\psi_G(e_1) = v_1v_2,\ \psi_G(e_2) = v_2v_3,\ \psi_G(e_3) = v_3v_3,\ \psi_G(e_4) = v_3v_4$$

$$\psi_G(e_5) = v_2v_4,\ \psi_G(e_6) = v_4v_5,\ \psi_G(e_7) = v_2v_5,\ \psi_G(e_8) = v_2v_5$$

Graphs and simple graphs

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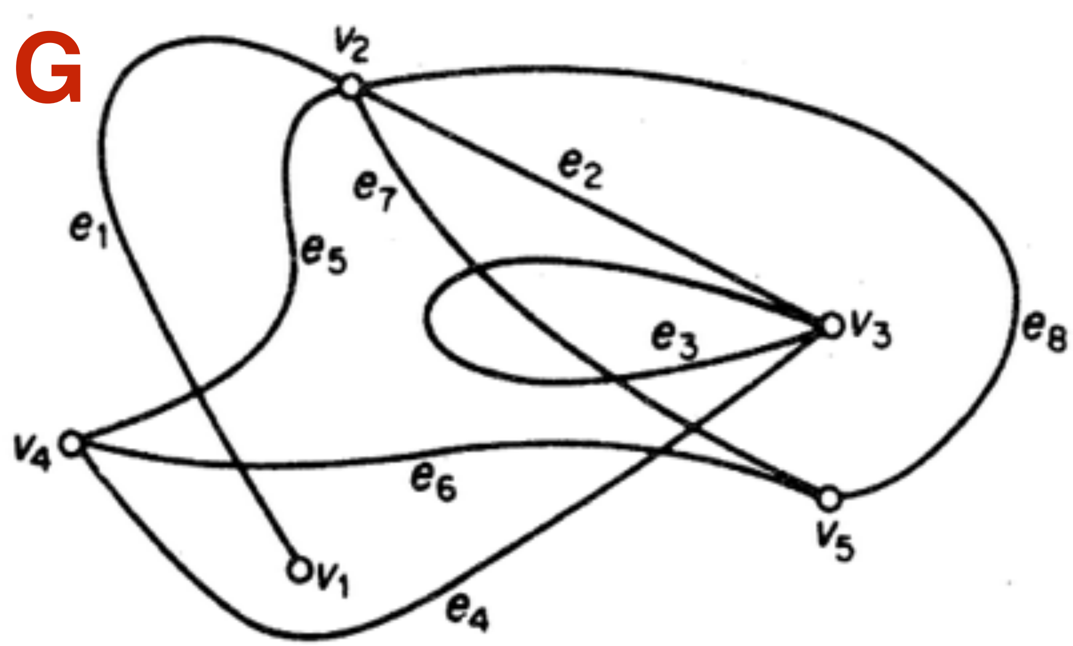
G



Graphs and simple graphs

Example 2

$$H = (V(H), E(H), \psi_H)$$

where

$$V(H) = \{u, v, w, x, y\}$$
$$E(H) = \{a, b, c, d, e, f, g, h\}$$

and ψ_H is defined by

$$\psi_H(a) = uv, \quad \psi_H(b) = uu, \quad \psi_H(c) = vw, \quad \psi_H(d) = wx$$
$$\psi_H(e) = vx, \quad \psi_H(f) = wx, \quad \psi_H(g) = ux, \quad \psi_H(h) = xy$$

Graphs and simple graphs

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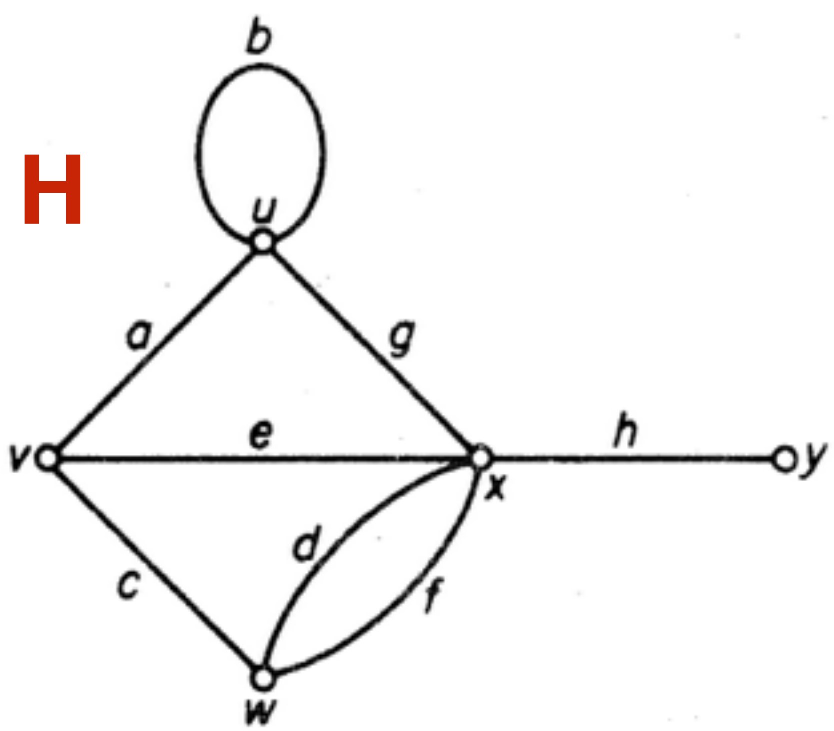

Graphs and simple graphs

If e is an edge and u and v are vertices such that $\psi_G(e) = uv$, then e is said to *join* u and v

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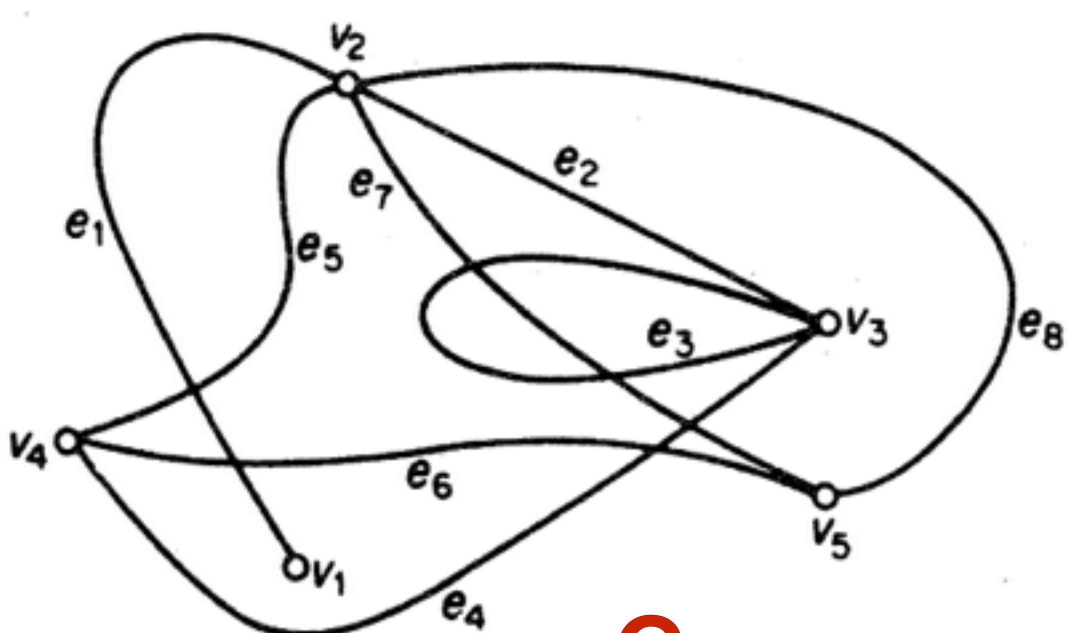
Graphs are so named because they can be represented graphically

There is no unique way of drawing a graph; the relative positions of points representing vertices and lines representing edges have no significance.



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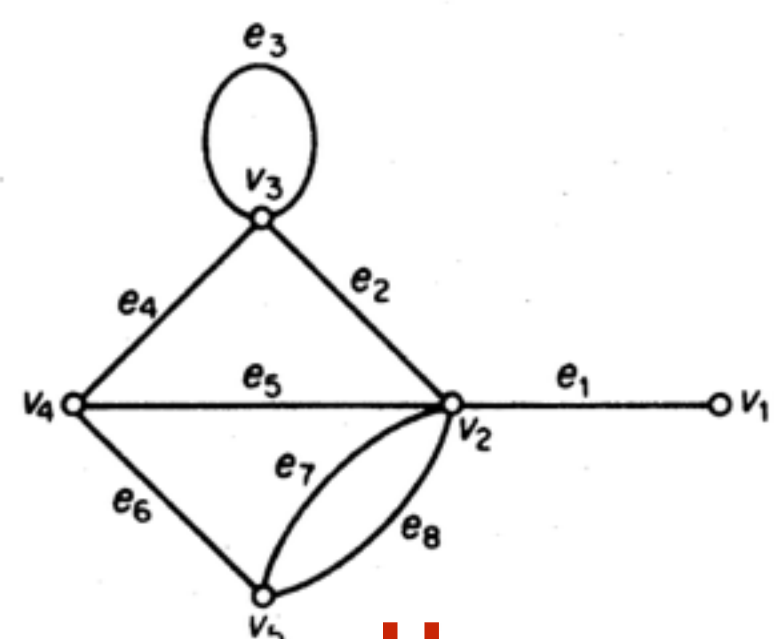
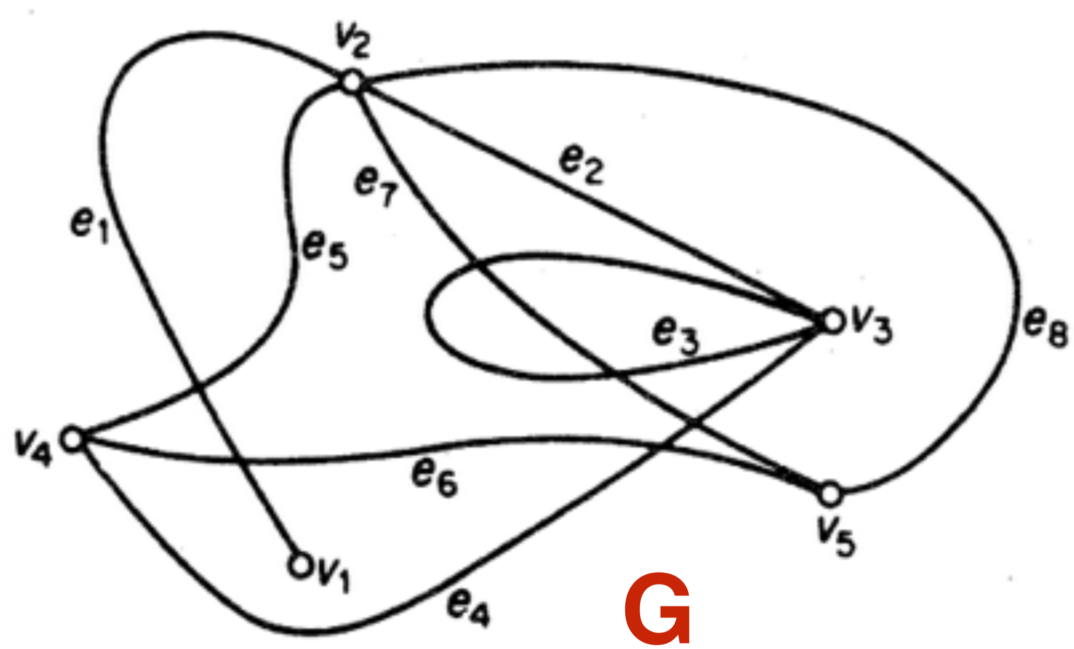


H

Graphs and simple graphs

Most of the definitions and concepts in graph theory are suggested by the graphical representation. The ends of an edge are said to be *incident* with the edge, and vice versa. Two vertices which are incident with a common edge are *adjacent*, as are two edges which are incident with a common vertex. An edge with identical ends is called a *loop*, and an edge with distinct ends a *link*. For example, the edge e_3 of G (figure 1.2) is a loop; all other edges of G are links.



Graphical representation: Planar Graphs

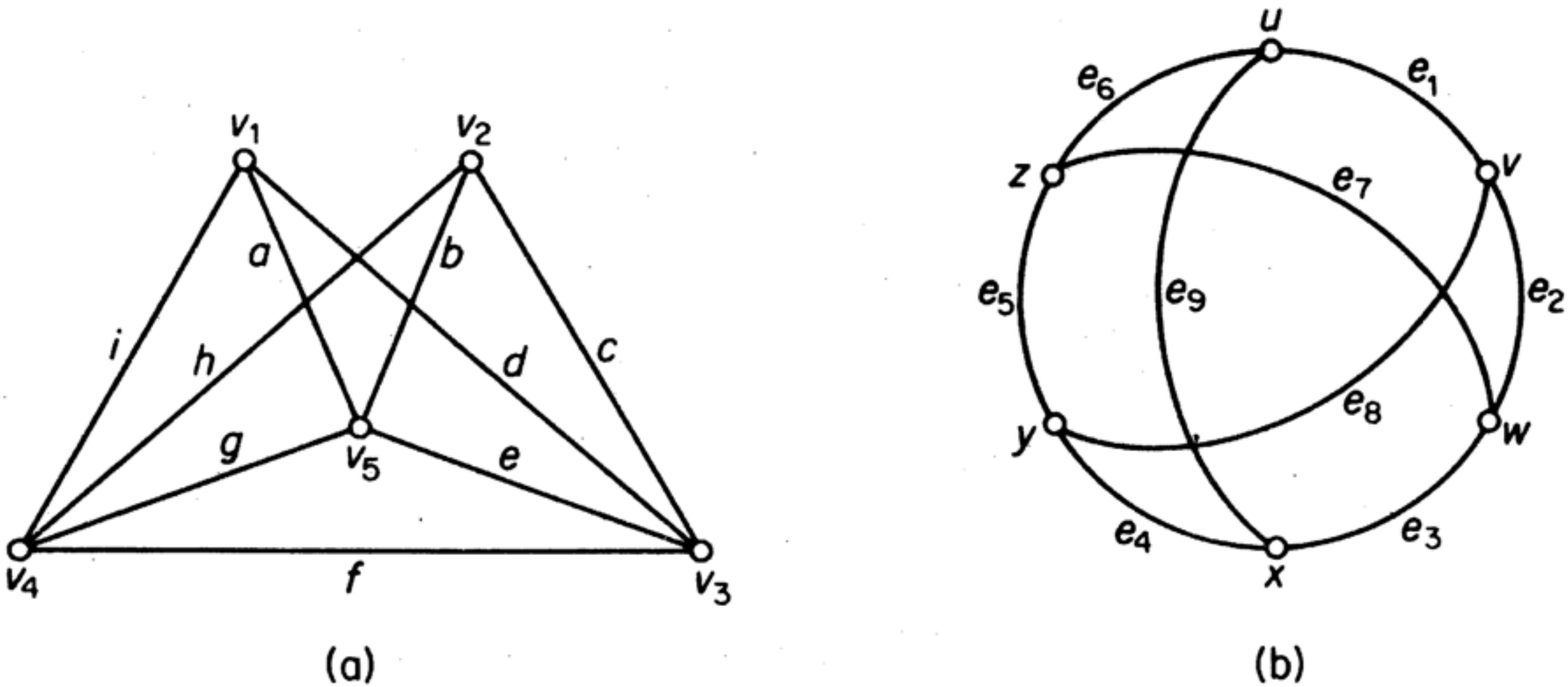


Figure 1.3. Planar and nonplanar graphs

Graphs and simple graphs

A graph is *finite* if both its vertex set and edge set are finite. In this book we study only finite graphs, and so the term 'graph' always means 'finite graph'. We call a graph with just one vertex *trivial* and all other graphs *nontrivial*.

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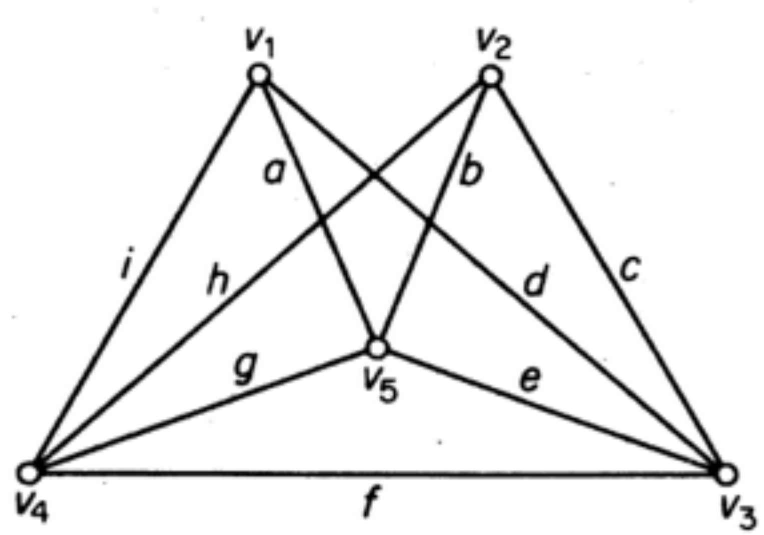
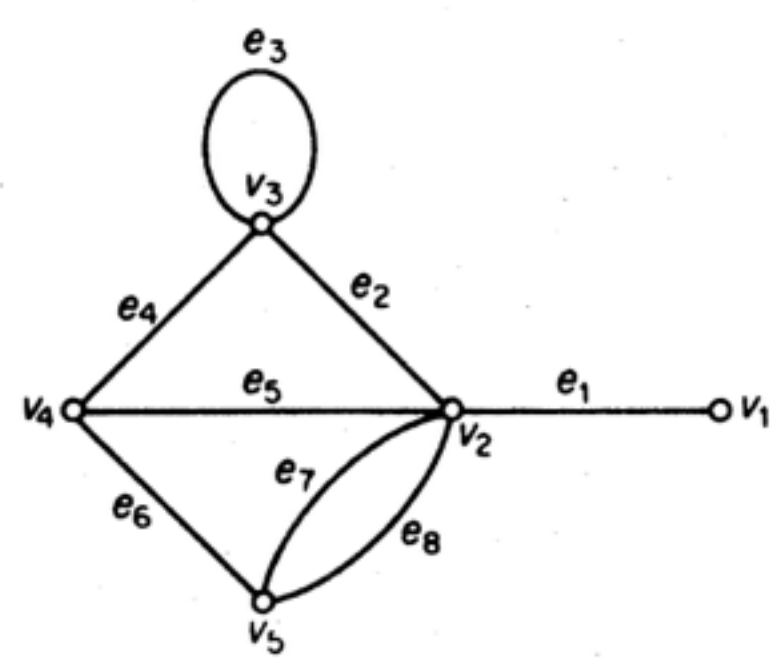
A graph is *simple* if it has no loops and no two of its links join the same pair of vertices. The graphs of figure 1.1 are not simple, whereas the graphs of figure 1.3 are. Much of graph theory is concerned with the study of simple graphs





Graphs and simple graphs

We use the symbols $\nu(G)$ and $\varepsilon(G)$ to denote the numbers of vertices and edges in graph G. Throughout the book the letter G denotes a graph. Moreover, when just one graph is under discussion, we usually denote this graph by G. We then omit the letter G from graph-theoretic symbols and write, for instance, V, E, ν and ε instead of $V(G)$, $E(G)$, $\nu(G)$ and $\varepsilon(G)$.

Graphs and simple graphs

We use the symbols **n(G) and m(G)** to denote the numbers of vertices and edges in graph G. Throughout the book the letter G denotes a graph. Moreover, when just one graph is under discussion, we usually denote this graph by G. We then omit the letter G from graph-theoretic svmbols and write, for instance, V, E, **n and m** instead of $V(G)$, $E(G)$, **n(G) and m(G)**.

Graphs and simple graphs

Exercises:

Draw a different diagram of the graph of figure 1.3*a* to show that it is indeed planar.

Show that if G is simple, then $\varepsilon \leq \binom{\nu}{2}$.

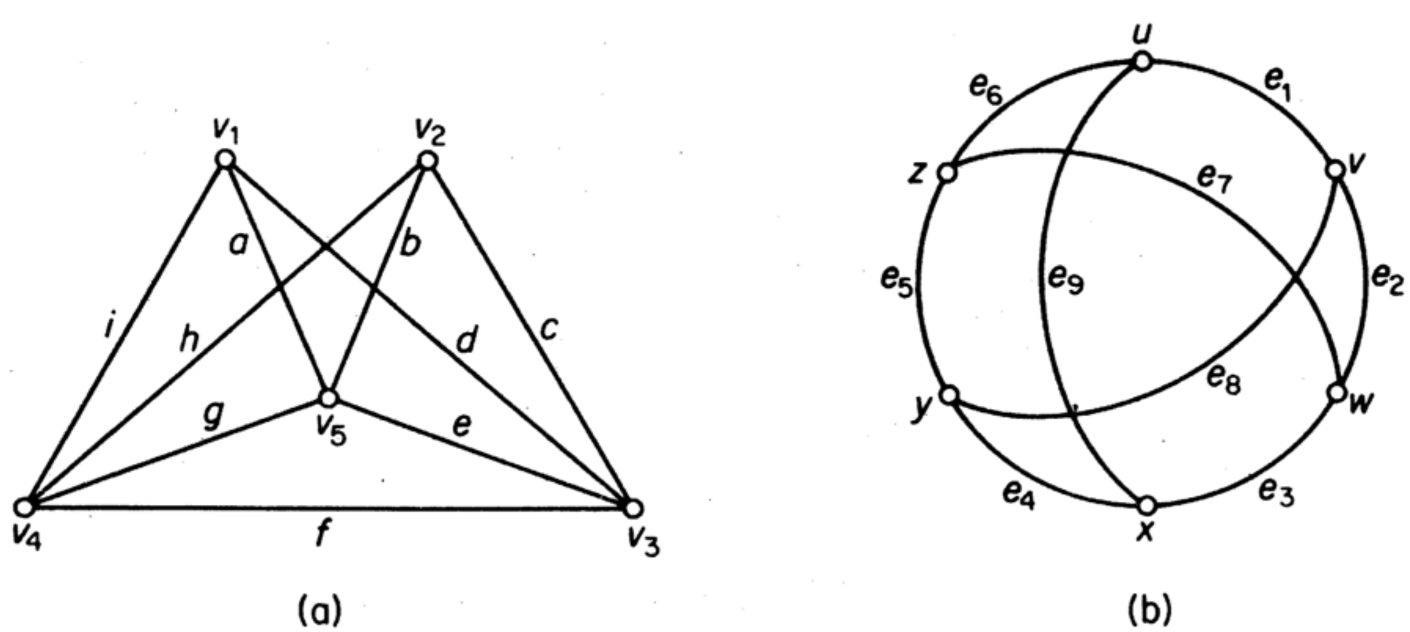
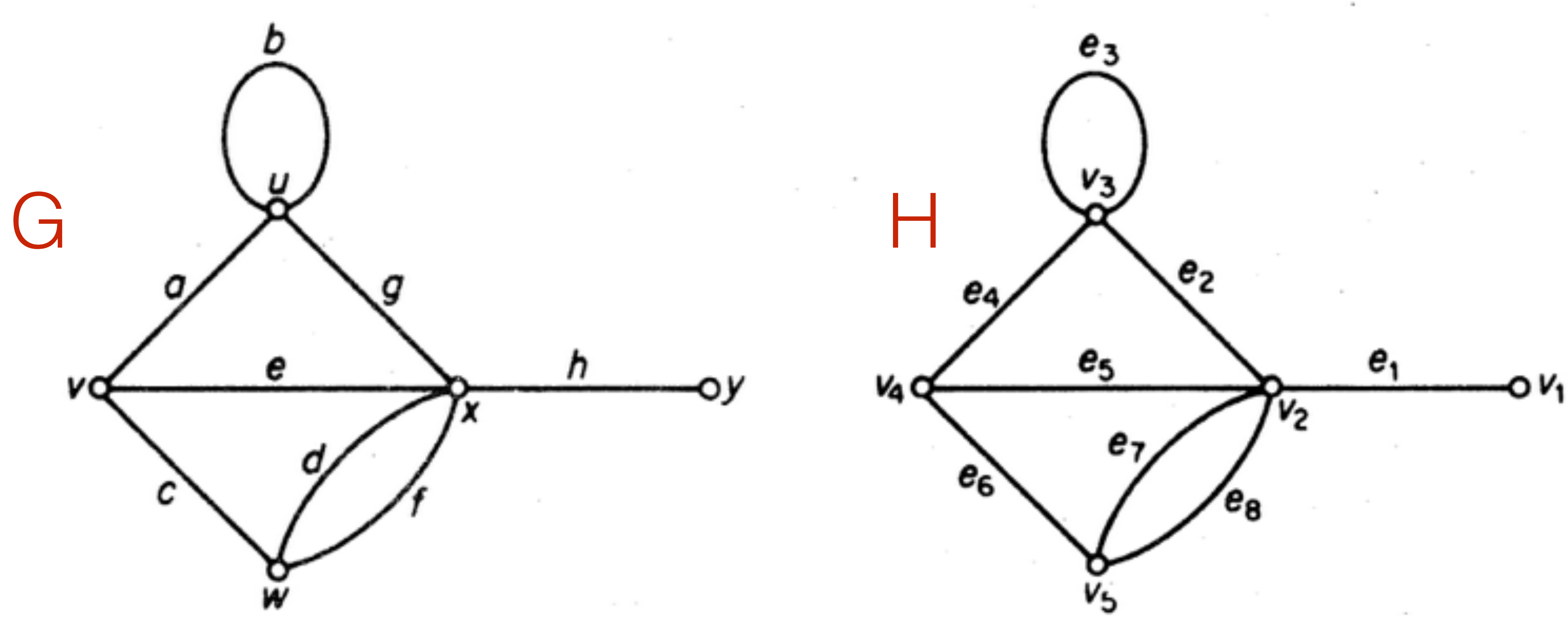


Figure 1.3. Planar and nonplanar graphs

Graph Isomorphism

Two graphs G and H are *identical* (written $G = H$) if $V(G) = V(H)$, $E(G) = E(H)$, and $\psi_G = \psi_H$. If two graphs are identical then they can clearly be represented by identical diagrams.

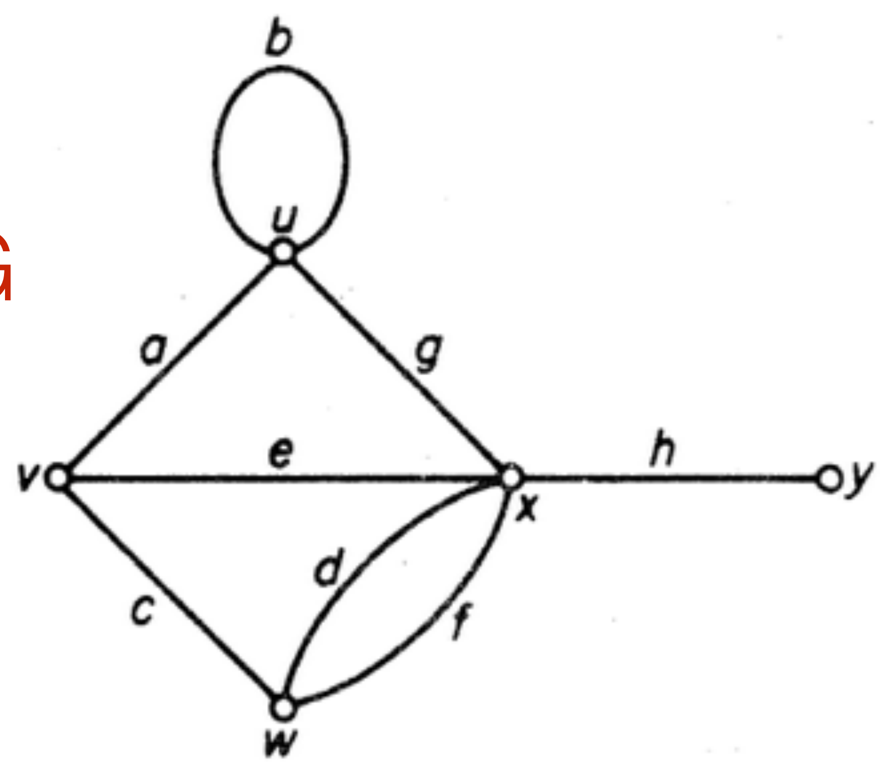
However, it is also possible for graphs that are not identical to have essentially the same diagram. For example, the diagrams of G in figure 1.2 and H in figure 1.1 look exactly the same, with the exception that their vertices and edges have different labels.



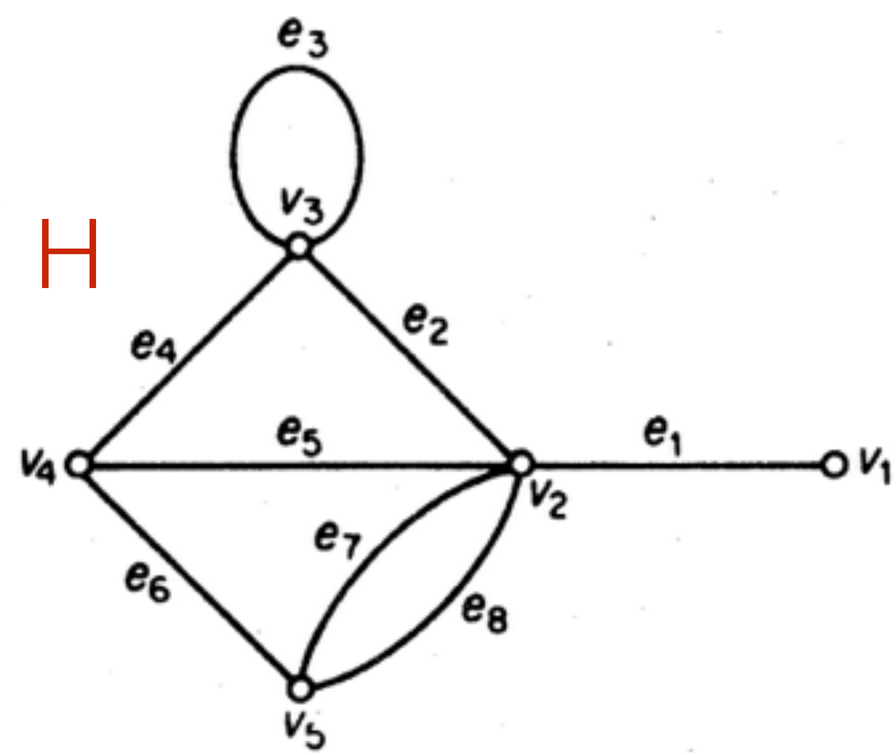
Graph Isomorphism

G and H are not identical, but isomorphic.

G



H

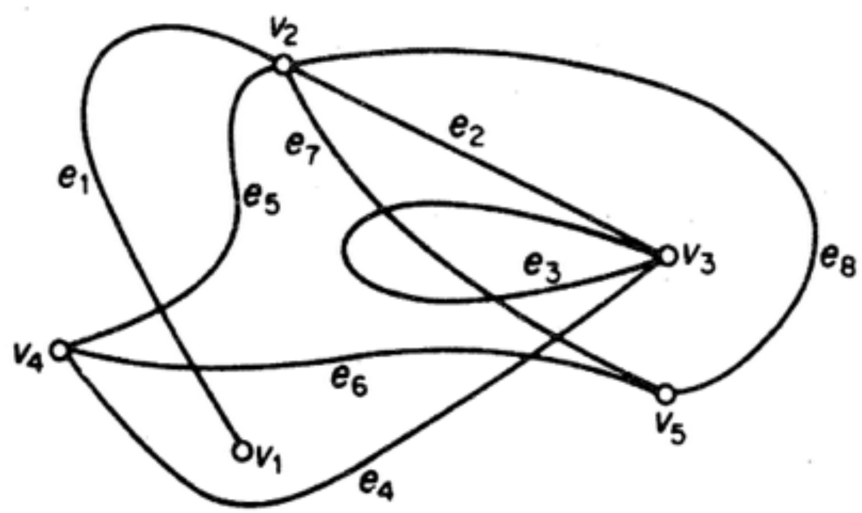


Graph Isomorphism

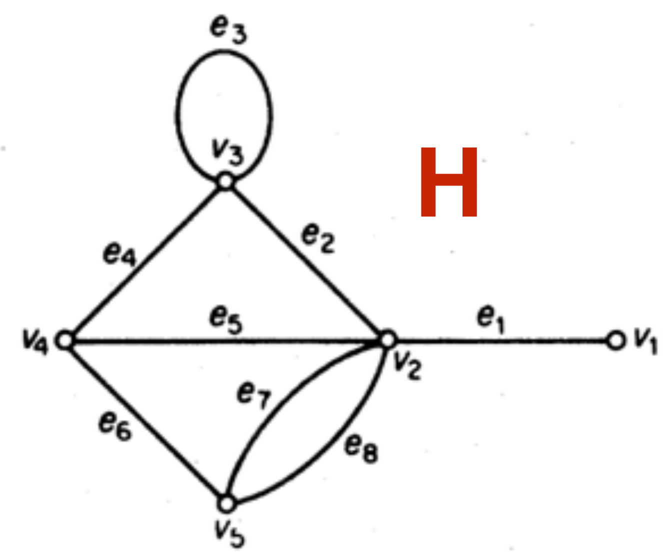
In general, two graphs G and H are said to be *isomorphic* (written $G \cong H$) if there are bijections $\theta : V(G) \to V(H)$ and $\phi : E(G) \to E(H)$ such that $\psi_G(e) = uv$ if and only if $\psi_H(\phi(e)) = \theta(u)\theta(v)$; such a pair (θ, ϕ) of mappings is called an *isomorphism* between G and H.

Graph Isomorphism

G



H



To show that two graphs are isomorphic, one must indicate an isomorphism between them. The pair of mappings (θ, ϕ) defined by

$$\theta(v_1) = y, \qquad \theta(v_2) = x, \qquad \theta(v_3) = u, \qquad \theta(v_4) = v, \qquad \theta(v_5) = w$$

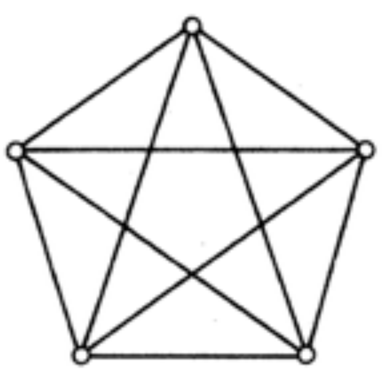
and

$$\phi(e_1) = h, \qquad \phi(e_2) = g, \qquad \phi(e_3) = b, \qquad \phi(e_4) = a$$
$$\phi(e_5) = e, \qquad \phi(e_6) = c, \qquad \phi(e_7) = d, \qquad \phi(e_8) = f$$

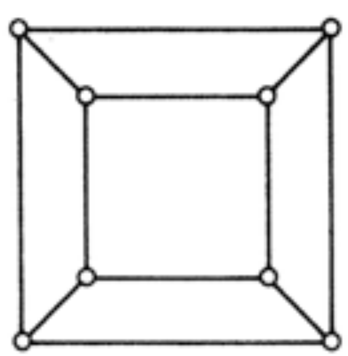
is an isomorphism between the graphs G and H of examples 1 and 2; G and H clearly have the same structure, and differ only in the names of vertices and edges.

Graph Isomorphism

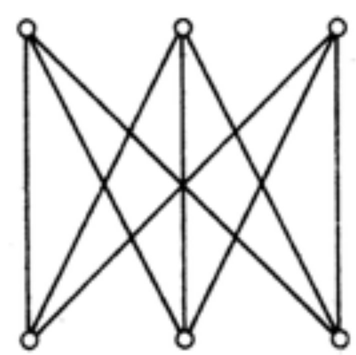
We conclude this section by introducing some special classes of graphs. A simple graph in which each pair of distinct vertices is joined by an edge is called a *complete graph*. Up to isomorphism, there is just one complete graph on n vertices; it is denoted by K_n. A drawing of K_5 is shown in figure 1.4*a*. An *empty graph*, on the other hand, is one with no edges. A *bipartite graph* is one whose vertex set can be partitioned into two subsets X and Y, so that each edge has one end in X and one end in Y; such a partition (X, Y) is called a *bipartition* of the graph. A *complete bipartite graph* is a simple bipartite graph with bipartition (X, Y) in which each vertex of X is joined to each vertex of Y; if $|X| = m$ and $|Y| = n$, such a graph is denoted by $K_{m,n}$. The graph defined by the vertices and edges of a cube (figure 1.4*b*) is bipartite; the graph in figure 1.4*c* is the complete bipartite graph $K_{3,3}$.



(a)



(b)



(c)

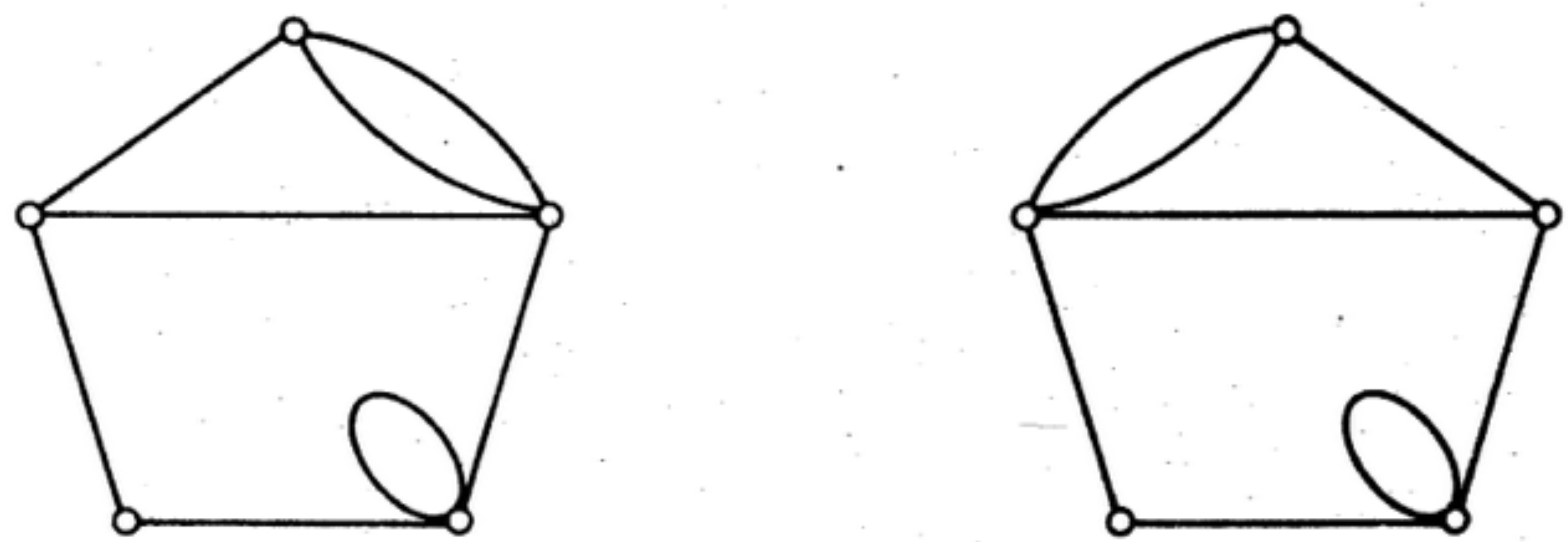
Figure 1.4. (*a*) K_5; (*b*) the cube; (*c*) $K_{3,3}$

Exercises

1 Find an isomorphism between the graphs G and H of examples 1 and 2 different from the one given.

2 (*a*) Show that if $G \cong H$, then $\nu(G) = \nu(H)$ and $\varepsilon(G) = \varepsilon(H)$.
(*b*) Give an example to show that the converse is false.

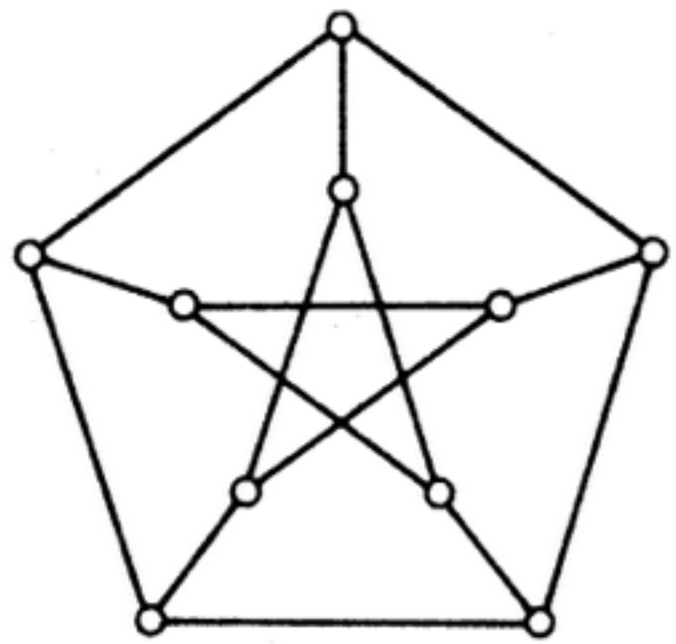
3 Show that the following graphs are not isomorphic:

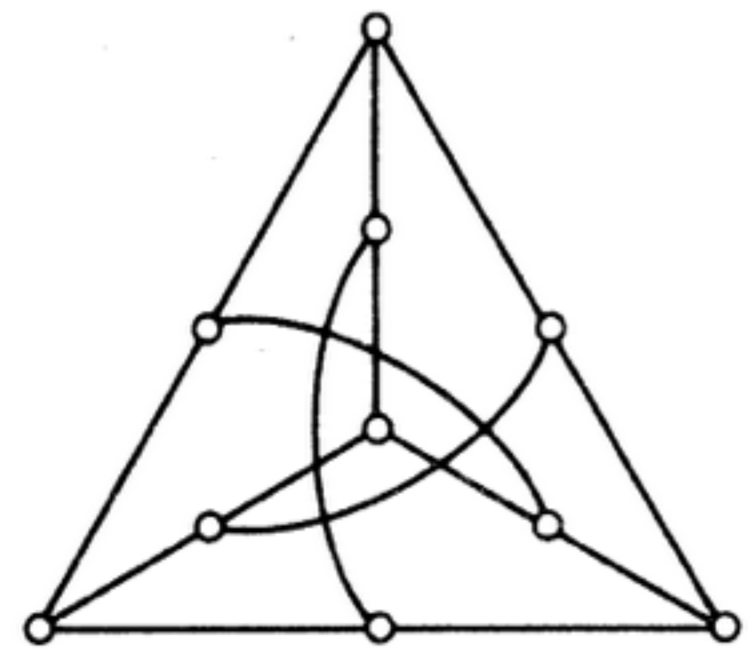


4 Show that there are eleven nonisomorphic simple graphs on four vertices.

5 Show that two simple graphs G and H are isomorphic if and only if there is a bijection $\theta : V(G) \rightarrow V(H)$ such that $uv \in E(G)$ if and only if $\theta(u)\theta(v) \in E(H)$.

6 Show that the following graphs are isomorphic:





7 Let G be simple. Show that $\varepsilon = \binom{\nu}{2}$ if and only if G is complete.

8 Show that

(*a*) $\varepsilon(K_{m,n}) = mn$;

(*b*) if G is simple and bipartite, then $\varepsilon \leq \nu^2/4$.

10 The *k-cube* is the graph whose vertices are the ordered k-tuples of 0's and 1's, two vertices being joined if and only if they differ in exactly one coordinate. (The graph shown in figure 1.4*b* is just the 3-cube.) Show that the k-cube has 2^k vertices, $k2^{k-1}$ edges and is bipartite.

11 (*a*) The *complement* G^c of a simple graph G is the simple graph with vertex set V, two vertices being adjacent in G^c if and only if they are not adjacent in G. Describe the graphs K_n^c and $K_{m,n}^c$.

(*b*) A simple graph G is *self-complementary* if $G \cong G^c$. Show that if G is self-complementary, then $\nu \equiv 0$, 1 (mod 4).