On Partitions and Relations

Paulo A. S. Veloso

COPPE-UFRJ / Progr. Eng. Sist. Comput.

pasveloso@gmail.com

Abstract: We examine some concepts and problems about partitions and relations suggested by J.-B. Joinet.

Research partly sponsored by CNPq

1 Introduction

Partitions and Relations

1. Equivalence relations connected to partitions (well known)

 ($\rightharpoonup$) Every equivalence relation gives a partition.

 ($\leftharpoondown$) Every partition comes from an equivalence relation.

2. Jean-Baptiste Joinet

 (¡) Some non-equivalence relations give partitions !

 Homo hominis lupus Virus feminæ lupus

 (¿) Which non-equivalence relations give partitions ?

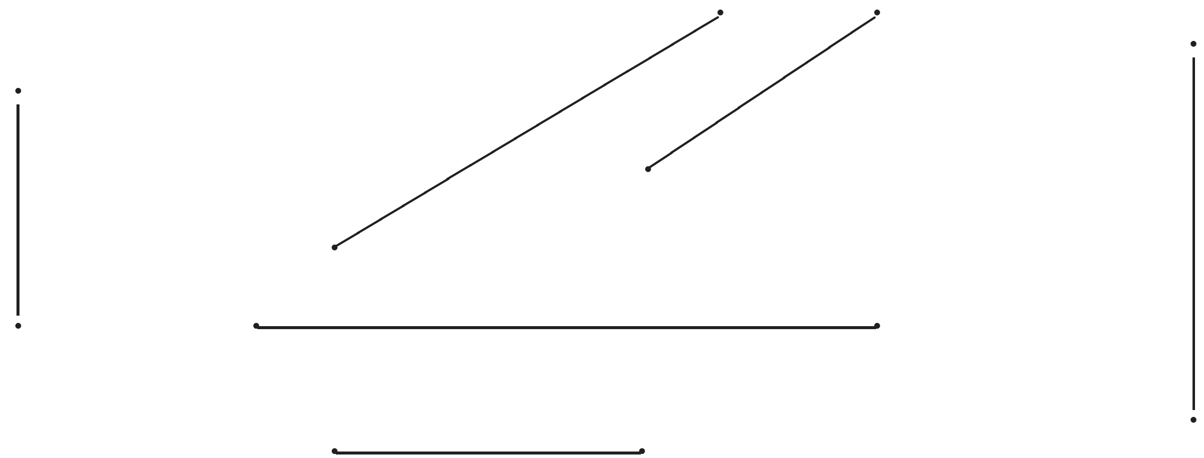
OUTLINE

2 Partitions and Relations

2.1 Partitions

Importance abstraction

Example 2.1 (Geometry)**.** *Direction (inclination)*



Direction $:=$ *set of (parallel) straight lines*.

♭

Example 2.2 (Arithmetic)**.** *Rationals and fractions*

Fraction: $\dfrac{\textit{Numerator}}{\textit{Denominator}}$

$$\frac{1}{6} + \frac{1}{10} = \frac{4}{15}$$

$$\left.\begin{array}{ccc} \dfrac{1}{6} & \sim & \dfrac{10}{60} \\[2em] \dfrac{1}{10} & \sim & \dfrac{6}{60} \end{array}\right\} \overset{+}{=} \frac{16}{60} \sim \frac{4}{15}$$

Several fractions for a single rational.

Rational $:=$ *set of fractions*.

♭

Partition of set S

set $\mathtt{P}$ of subsets of S (called *blocks*), s. t.:

($\not\emptyset$) $\forall B \in \mathtt{P}: \; B \neq \emptyset$ non-void

every block has some element

($\bigcup$) $S \subseteq \bigcup_{B \in \mathtt{P}} B$ cover

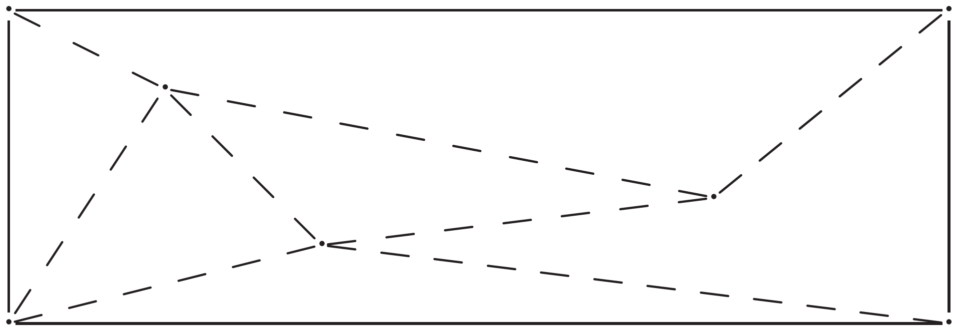
every element is in some block

($\cap$) $\forall B, C \in \mathtt{P} : B \cap C \neq \emptyset \;\Rightarrow\; B = C$ disjoint

distinct blocks are disjoint

∂

A partition looks as follows:



(See Examples 2.3: Extreme partitions of set ℕ of naturals, p. 8 and 2.4: Particular partitions, p. 9.)

Example 2.3 (Extreme partitions of set $\mathbb{N}$ of naturals)**.**

One-block partition *Singleton-block partition*

$$\{\underbrace{\{0,1,2,\ldots,n,\ldots\}}_{\mathbb{N}}\} \qquad \{\{0\},\{1\},\{2\},\ldots,\{n\},\ldots\}$$

$\flat$

Remark 2.1 (Extreme partitions)**.** *Non-empty set* $\mathsf{S} \neq \emptyset$.

- One-block partition $\mathtt{Q} \coloneqq \{\mathsf{S}\}$
- Partition into singleton blocks $\mathtt{R} \coloneqq \{\{\mathsf{a}\} \subseteq \mathsf{S} \,/\, \mathsf{a} \in \mathsf{S}\}$

$\surd$

Example 2.4 (Particular partitions). *Partitions of some sets.*

- *Set* $\{a,b,c,d,e\}$ *partition* P_3

$$\underbrace{\left\{\begin{matrix} a \\ d \end{matrix}\right\}}_{B_0} \qquad \underbrace{\left\{\begin{matrix} b \\ e \end{matrix}\right\}}_{B_1} \qquad \underbrace{\{\ c\ \}}_{B_2}$$

P_3: *finitely many (3)* *finite blocks*

B_0	B_1	B_2
{*Ana* , *Diogo*}	{*Beta* , *Edu*}	{*Ciça*}
{*Abel* , *Deportivo*}	{*Beto* , *Excelsior*}	{*Cadu*}
$\{0,3\}$	$\{1,4\}$	$\{2\}$

- *Set $\mathbb{N}$ of naturals* *partition $\mathtt{P}_4$ (modulo 4)*

$$\underbrace{\left\{\begin{array}{c}0\\4\\\vdots\\4\cdot n\\\vdots\end{array}\right\}}_{C_0}\quad\underbrace{\left\{\begin{array}{c}1\\5\\\vdots\\4\cdot n+1\\\vdots\end{array}\right\}}_{C_1}\quad\underbrace{\left\{\begin{array}{c}2\\6\\\vdots\\4\cdot n+2\\\vdots\end{array}\right\}}_{C_2}\quad\underbrace{\left\{\begin{array}{c}3\\7\\\vdots\\4\cdot n+3\\\vdots\end{array}\right\}}_{C_3}$$

Block C_r: *remainder r* *division by 4*

$\mathtt{P}_4$: *finitely many (4)* *infinite blocks*

- *Set $\mathbb{Z}$ of integers* *partition* $\mathtt{P}_{||}$ *(absolute-value)*

$$\underbrace{\left\{ 0 \right\}}_{D_0} \quad \underbrace{\left\{ \begin{array}{c} +1 \\ -1 \end{array} \right\}}_{D_1} \quad \underbrace{\left\{ \begin{array}{c} +2 \\ -2 \end{array} \right\}}_{D_2} \quad \ldots \quad \underbrace{\left\{ \begin{array}{c} +n \\ -n \end{array} \right\}}_{D_n} \quad \ldots$$

$\mathtt{P}_{||}$: *infinitely many* *finite blocks*

D_0	D_1	D_2	D_3	...	D_n	...
$\{0\}$	$\{1, i\}$	$\{2, 2 \cdot i\}$	$\{3, 3 \cdot i\}$	...	$\{n, n \cdot i\}$	...
$\{0\}$	$\{+1, -1\}$	$\{+\frac{1}{2}, -\frac{1}{2}\}$	$\{+\frac{1}{3}, -\frac{1}{3}\}$	...	$\{+\frac{1}{n}, -\frac{1}{n}\}$	...
$\{0\}$	$\{+\pi, -\pi\}$	$\{+\frac{\pi}{2}, -\frac{\pi}{2}\}$	$\{+\frac{\pi}{3}, -\frac{\pi}{3}\}$	...	$\{+\frac{\pi}{n}, -\frac{\pi}{n}\}$	...

- *Set* $\mathbb{N}_+$ *of positive naturals* *partition* $\mathtt{P}_\infty$

$$\underbrace{\left\{\begin{array}{c} 1 \\ 3 \\ 5 \\ 7 \\ \vdots \\ (2\cdot m+1) \\ \vdots \end{array}\right\}}_{E_0} \quad \underbrace{\left\{\begin{array}{c} 2 \\ 6 \\ 10 \\ 14 \\ \vdots \\ 2\cdot(2\cdot m+1) \\ \vdots \end{array}\right\}}_{E_1} \quad \cdots \quad \underbrace{\left\{\begin{array}{c} 2^n \\ 2^n\cdot 3 \\ 2^n\cdot 5 \\ 2^n\cdot 7 \\ \vdots \\ 2^n\cdot(2\cdot m+1) \\ \vdots \end{array}\right\}}_{E_n} \quad \cdots$$

$\mathtt{P}_\infty$: *infinitely many* *infinite blocks*

Partition	*Set*	*Nbr. blocks*	*Block size*
P_3	$\{a, b, c, d, e\}$	*finite*	*finite*
P_4	$\mathbb{N}$	*finite*	*infinite*
$P_{\|\|}$	$\mathbb{Z}$	*infinite*	*finite*
P_∞	$\mathbb{N}_+$	*infinite*	*infinite*

♭

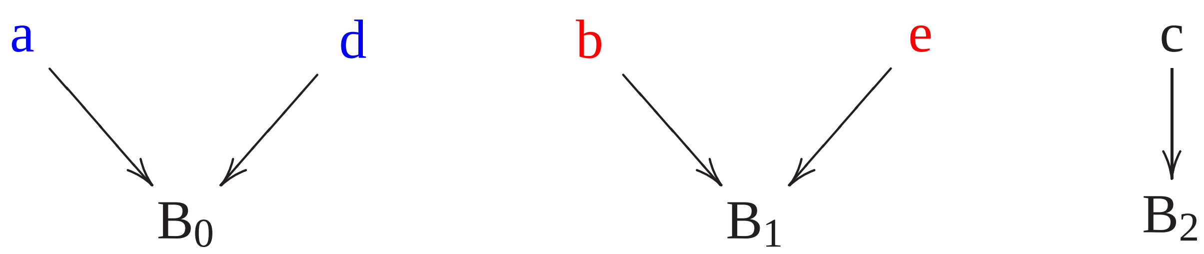
Remark 2.2 (Partition block)**.** *Unique block with element.*
Partition $\mathtt{P}$ *of* S*, element* $\mathsf{s} \in \mathsf{S}$*:* $\exists !$ *block* $\mathsf{B} \in \mathtt{P}$*, s. t.* $\mathsf{s} \in \mathsf{B}$. √

Partition block function: $\mathtt{P}(\bullet) : \mathsf{S} \twoheadrightarrow \mathtt{P}$

$$\forall \mathsf{s} \in \mathsf{S}\ \forall \mathsf{B} \in \mathtt{P} : \Big(\mathtt{P}(\mathsf{s}) = \mathsf{B} \quad \Leftrightarrow \quad \mathsf{s} \in \mathsf{B} \Big)$$

∂

Example 2.5 (Partition block function)**.** *Partition* $\mathtt{P}_3$
(Example 2.4: Particular partitions, p. 9), function $\mathtt{P}_3(\bullet)$*:*



♭

2.2 Relations

Relation $\longrightarrow$ on set S (See Example 2.3: Classes of relations, p. 8.)

($\mathtt{Dmn}$) *Domain* — elements with successor

$$\mathtt{Dmn}(\longrightarrow) \;\coloneqq\; \{\, a \in S \,/\, \exists b \in S : a \;\longrightarrow\; b \,\}$$

($\mathtt{Img}$) *Image* — elements with predecessor

$$\mathtt{Img}(\longrightarrow) \;\coloneqq\; \{\, b \in S \,/\, \exists a \in S : a \;\longrightarrow\; b \,\}$$

($[\,]$) *Class* of $a \in S$ — reached elements

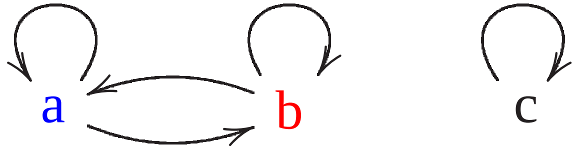
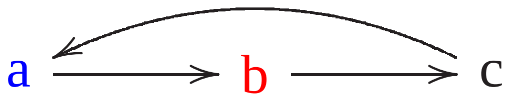
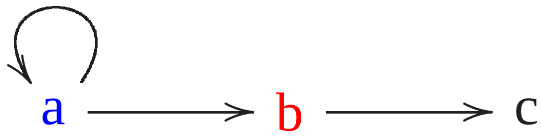
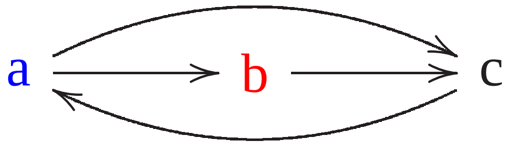
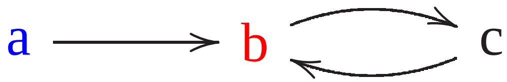
$$[a] \;\coloneqq\; \{\, b \in S \,/\, a \;\longrightarrow\; b \,\}$$

($/$) *Quotient set* — set of classes

$$S/\!\longrightarrow \;\coloneqq\; \{\, [s] \subseteq S \,/\, s \in S \,\}$$

∂

Example 2.6 (Classes of relations). *Set* $S = \{a, b, c\}$

Relation $\longrightarrow$	$[a]$	$[b]$	$[c]$	*Partition?*
	$\{a, b\}$	$\{a, b\}$	$\{c\}$	*Yes*
	$\{b\}$	$\{c\}$	$\{a\}$	*Yes*
	$\{a, b\}$	$\{c\}$	$\emptyset$	*No*
	$\{b, c\}$	$\{c\}$	$\{a\}$	*No*
	$\{b\}$	$\{c\}$	$\{b\}$	*No*

Properties of relations — relation $\longrightarrow$ on set S

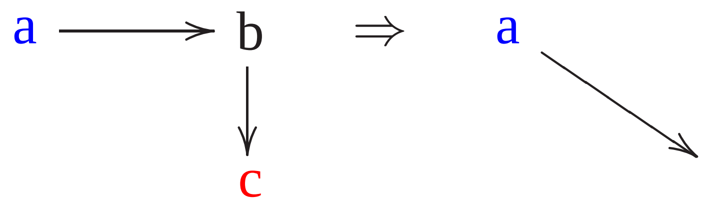
1. *Reflection point* $\quad s \circlearrowright$

   $\longrightarrow$ *reflexive* (`Rfl`) iff $\forall s \in S : s \longrightarrow s$ i. e. $\forall s \in S : s \in [s]$.

2. *Symmetric pair* $\quad a \curvearrowright b \;\Rightarrow\; a \curvearrowleft b$

   $\longrightarrow$ *symmetric* (`Smm`) iff $\forall a, b \in S : \; a \longrightarrow b \;\Rightarrow\; b \longrightarrow a$.

3. *Transitive triple*



   $\longrightarrow$ *transitive* (`Trn`) iff $\forall a, b, c \in S : \; a \longrightarrow b \longrightarrow c \;\Rightarrow\; a \longrightarrow c$.

4. *Equivalence* (`Eqv`): reflexive, symmetric & transitive.

∂

(See Example 2.7: Relation modulo *k*, p. 18.)

Example 2.7 (Relation modulo k)**.** *Given natural $k \in \mathbb{N}$.*

1. *Relation* modulo k *on* $\mathbb{N}$: $\quad m \approx_k n \quad$ *iff* $\quad m - n \quad$ *is multiple of* k.

2. *For all* $m, n \in \mathbb{N}$: $\quad m \approx_1 n \quad$ & $\quad m \approx_0 n \;\Leftrightarrow\; m = n$.

3. *Example 2.4 (Particular partitions, p. 10) shows quotient* $\mathbb{N}/{\approx_4}$.

4. *Example 2.3 (Extreme partitions of set* $\mathbb{N}$ *of naturals, p. 8) shows quotients* $\mathbb{N}/{\approx_1}$ *(one-block) and* $\mathbb{N}/{\approx_0}$ *(singleton-blocks).*

5. *For each* $k \in \mathbb{N}$, $\approx_k$ *is an equivalence.*

♭

Remark 2.3 (Equivalences and partitions)**.** *Set* $\mathrm{S} \neq \emptyset$.

($\rightharpoonup$) *For each equivalence* $\sim$ *on* S*:* $\mathrm{S}/\!\sim$ *is a partition of* S.

($\leftharpoondown$) *For each partition of* $\mathtt{P}$ *of* S*: there is an equivalence* $\sim$ *on* S*, s. t.*
$\mathtt{P} = \mathrm{S}/\!\sim$ *(namely:* $\sim$ *s. t.* $\mathrm{a} \sim \mathrm{b}$ *iff* $\mathtt{P}(\mathrm{a}) = \mathtt{P}(\mathrm{b})$*).*

$\surd$

Natural question J.-B. Joinet

Question: Which partitions can be induced by a non-equivalence?

Conjecture: Every partition with more than 1 block.

2.3 Relations and Partitions

When is the quotient a partition? J.-B. Joinet

Proposition 2.1 (Quotient and partition)**.** *Given relation $\rightarrow$ on set $S \neq \emptyset$, quotient $S/\!\rightarrow$ is a partition iff $\rightarrow$ satisfies the 3 conditions:*

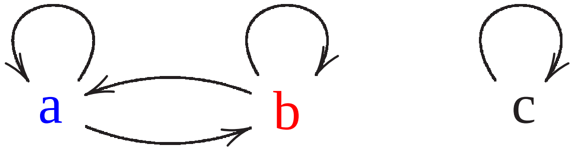
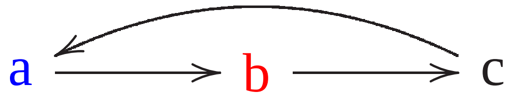
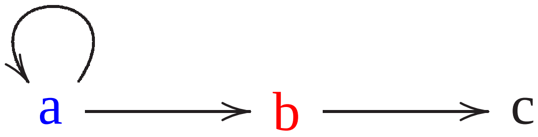
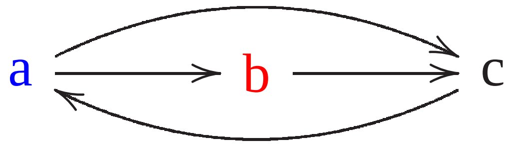
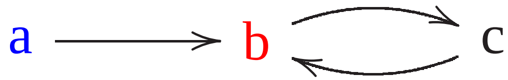
(δ) $S \subseteq \mathrm{Dmn}(\rightarrow)$ $\qquad \forall a \in S\ \exists b \in S : a \rightarrow b$

(ι) $S \subseteq \mathrm{Img}(\rightarrow)$ $\qquad \forall b \in S\ \exists a \in S : a \rightarrow b$

(γ) $$\exists s \in S \left(\begin{array}{ccc} b & \searrow & \\ & & s \\ c & \nearrow & \end{array}\right) \quad \Rightarrow \quad \forall t \in S \left(\begin{array}{ccc} b & \longrightarrow & t \\ & \Downarrow & \\ c & \longrightarrow & t \end{array}\right)$$

♮

Example 2.8 (Conditions for quotient partition). *Set* $S = \{a, b, c\}$.
Relations in Example 2.6 (Classes of relations, p. 16).

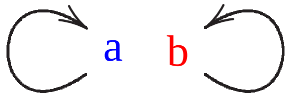
Relation →	(δ)	(ι)	(γ)	
	+	+	+	*Partition*
	+	+	+	*Partition*
	−	+	+	$[c] = \emptyset$
	+	+	−	$[a] \cap [b] \neq \emptyset$
	+	−	+	$a \notin [a] \cup [b] \cup [c]$

Reflexion points & confluence property (γ) (cf. p. 20)

Lemma 2.1 (Reflexive confluence)**.** *Relation $\rightarrow$ with confluence (γ)*

1. b*: reflexion point* 

$\Rightarrow$ (a, b, c)*: transitive triple*

2. a, b*: reflexion points* 

$\Rightarrow$ (a, b)*: symmetric pair*

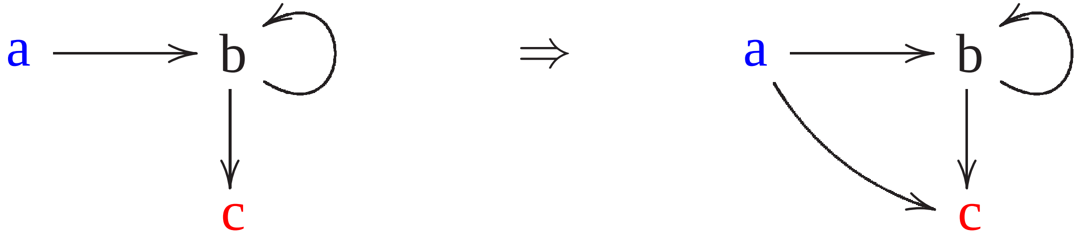
3. $\rightarrow$ *reflexive* $\Rightarrow$ $\begin{pmatrix} \rightarrow \textit{symmetric} \\ \rightarrow \textit{transitive} \end{pmatrix}$ $\therefore$ *equivalence*

♮

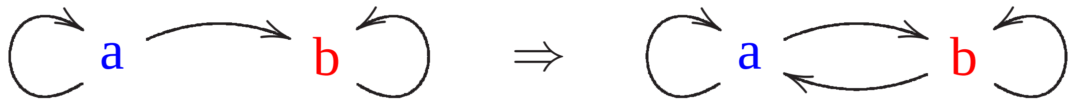
(See Example 2.9: Reflexion points and confluence, p. 23.)

Example 2.9 (Reflexion points and confluence). *Confluence property* (γ)

1. b: *reflexion point*



2. a, b: *reflexion points*



♭

Partition from non-equivalence

Corollary 2.1 (Partition from non-equivalence). *Relation* $\to$ *on* $S \neq \emptyset$

$$\begin{pmatrix} S/\!\to : \ \textit{partition} \\ \to \ \textit{non-equivalence} \end{pmatrix} \quad \Rightarrow \quad \to: \textit{non-reflexive.}$$

♮

Strong negative properties of relations relation $\rightarrow$ on set S

(siR) $\longrightarrow$ *strongly irreflexive* iff no reflexion point:

$$\forall s \in S : \ s \not\rightarrow s \qquad \text{i. e.} \qquad \forall s \in S : s \notin [s]$$

(saS) $\longrightarrow$ *strongly asymmetric* iff no symmetric pair:

$$\forall a, b \in S : \ a \longrightarrow b \ \Rightarrow \ b \not\rightarrow a$$

(saT) $\longrightarrow$ *strongly anti-transitive* iff no transitive triple:

$$\forall a, b, c \in S : \ a \longrightarrow b \longrightarrow c \ \Rightarrow \ a \not\rightarrow c$$

(saE) $\longrightarrow$ *strong anti-equivalence*:

strongly irreflexive, asymmetric & anti-transitive.

∂

(See Example 2.10: Relation *k*-successor, p. 25.)

Example 2.10 (Relation k-successor**).** *Given natural* $k \in \mathbb{N}$.

1. *Relation* k-successor *on* $\mathbb{Z}$*:* $m \prec_k n$ *iff* $m + k = n$.
2. *Relation* $\prec_1$ *on* $\mathbb{Z}$*:* *1 chain*

$$\dots \quad -2 \quad \prec_1 \quad -1 \quad \prec_1 \quad 0 \quad \prec_1 \quad +1 \quad \prec_1 \quad +2 \quad \prec_1 \quad \dots$$

3. *Relation* $\prec_2$ *on* $\mathbb{Z}$*:* *2 chains*

$$\dots \quad -4 \quad \prec_2 \quad -2 \quad \prec_2 \quad 0 \quad \prec_2 \quad +2 \quad \prec_2 \quad +4 \quad \dots$$

$$\dots \quad -3 \quad \prec_2 \quad - \quad \prec_2 \quad +1 \quad \prec_2 \quad +3 \quad \prec_2 \quad +5 \quad \dots$$

4. *Relation* $\prec_3$ *on* $\mathbb{Z}$*:* *3 chains*

$$\dots \quad -6 \quad \prec_3 \quad -3 \quad \prec_3 \quad 0 \quad \prec_3 \quad +3 \quad \prec_3 \quad +6 \quad \dots$$

$$\dots \quad -1 \quad \prec_3 \quad -2 \quad \prec_3 \quad +1 \quad \prec_3 \quad +4 \quad \prec_3 \quad +7 \quad \dots$$

$$\dots \quad -2 \quad \prec_3 \quad -1 \quad \prec_3 \quad +2 \quad \prec_3 \quad +5 \quad \prec_3 \quad +8 \quad \dots$$

5. *For each natural* $k > 0$, $\prec_k$ *is a strong anti-equivalence.* ♭

Outline

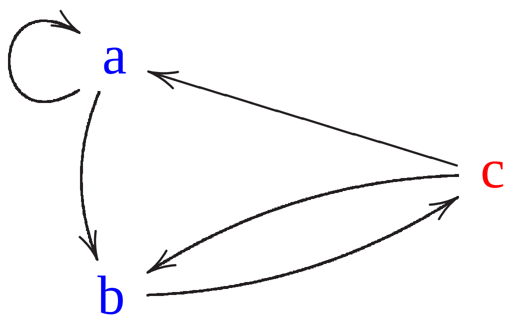
1. Introduction — situation
2. Partitions and Relations — basic definitions and results
   - 2.1. Partitions — abstraction, definition, examples
   - 2.2. Relations — class, quotient, equivalence, examples
   - 2.3. Relations and Partitions — condition: quotient partition
3. Relations for Partitions: Examples — finite and infinite partitions
   - 3.1. Relations for Finite Partitions — finite & infinite blocks
   - 3.2. Relations for Infinite Partitions — finite & infinite blocks
4. Relations for Partitions: Analysis — relations inducing partition
   - 4.1. Limitative Results — small partitions
   - 4.2. Relation from Partition — construction, properties
5. Relations for Non-trivial Partitions — results and construction
6. Relations Inducing Partitions — summary

3 Relations and Partitions: Examples

3.1 Relations for Finite Partitions

Example 3.1 (Partition with 2 finite blocks). *Set* $S = \{a,b,c\}$, *2-block partition* $P = \{B,C\}$, *with* $B = \{a,b\}$ & $C = \{c\}$.
Relations on S *inducing partition* P.

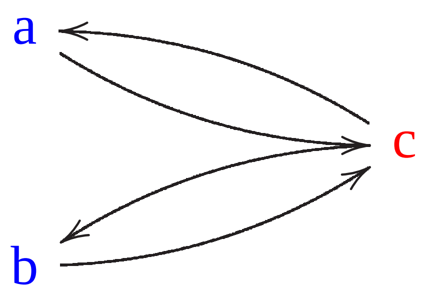
1. *Relation* $\rightarrow = \{(a,a),(a,b),(b,c),(c,a),(c,b)\}$:



$$[a] = [c] = \underbrace{\{a,b\}}_{B} \qquad [b] = \underbrace{\{c\}}_{C}$$

Relation $\rightarrow$ $\underbrace{\textit{non-reflexive}}_{b}$, $\underbrace{\textit{non-symmetric}}_{(a,b)}$, $\underbrace{\textit{non-transitive}}_{(b,c,b)}$.

2. *Relation* $\rightarrow = \{(a,c),(b,c),(c,a),(c,b)\}$:



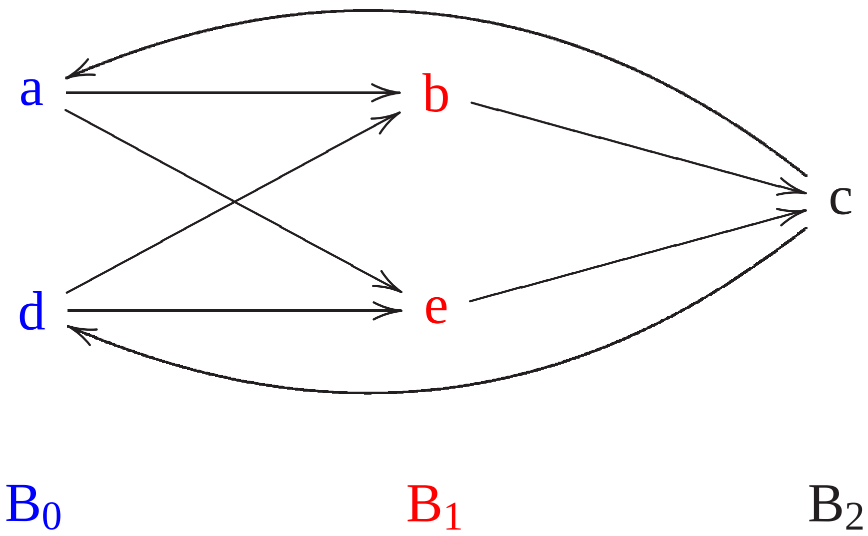
$$[a] = [b] = \underbrace{\{c\}}_{C} \qquad [c] = \underbrace{\{a,b\}}_{B}$$

Relation $\rightarrow$ *symmetric, non-transitive, strongly irreflexive.*

So, have non-equivalences inducing partition $\mathtt{P} = \{\{a,b\},\{c\}\}$.

Example 3.2 (Partition with 3 finite blocks). *Set* $\{a, b, c, d, e\}$,
$P_3 = \{B_0, B_1, B_2\}$, *with blocks* $B_0 = \{a, d\}$, $B_1 = \{b, e\}$ & $B_2 = \{c\}$.
(cf. Example 2.4: Particular partitions, p. 9).

Relation $\rightarrow$ *on* $\{a, b, c, d, e\}$*:*



$$[a] = [d] = \{b, e\} = B_1$$

$$[b] = [e] = \{c\} = B_2$$

$$[c] = \{a, d\} = B_0$$

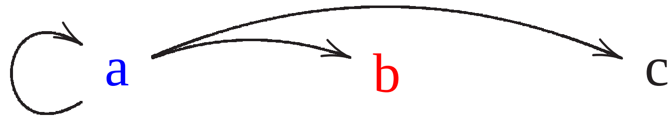
Relation $\rightarrow$ *strong anti-equivalence.*

♭

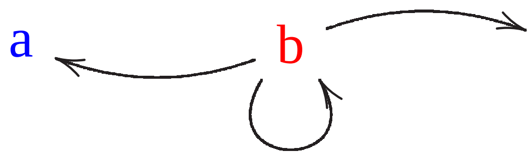
Example 3.3 (Single finite block)**.** *Set* $S = \{a, b, c\}$, *relation* $\rightarrow$ *on* S *inducing single-block partition* $P_0 = \{S\}$.

Relation $\rightarrow$ *on* $S = \{a, b, c\}$*:*

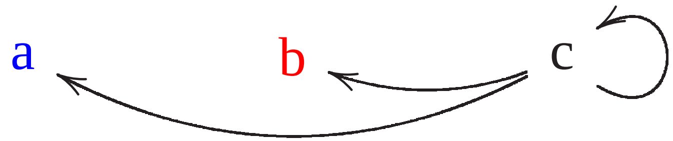
$[a] = \{a, b, c\} \quad \Rightarrow$



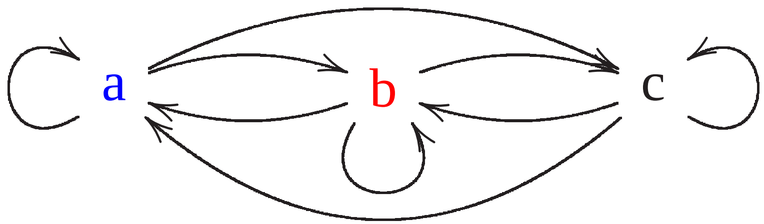
$[b] = \{a, b, c\} \quad \Rightarrow$



$[c] = \{a, b, c\} \quad \Rightarrow$



$\therefore$



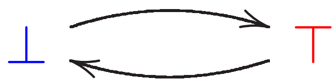
Relation $\rightarrow$ *(full) equivalence.* $\flat$

Example 3.4 (Two-singleton partition)**.** *Set* $\mathrm{S} = \{\perp, \top\}$*,*
2-block partition $\mathtt{P} = \{\{\perp\}, \{\top\}\}$*.*
Strongly irreflexive relation $\to$ *on* S *inducing* $\mathtt{P}$*. So,* $\perp \not\to \perp$ *&* $\top \not\to \top$*.*

Relation $\to$ *on* $\mathrm{S} = \{\perp, \top\}$*:*

$[\perp] \neq \emptyset \quad \Rightarrow$ 

$[\top] \neq \emptyset \quad \Rightarrow$ 

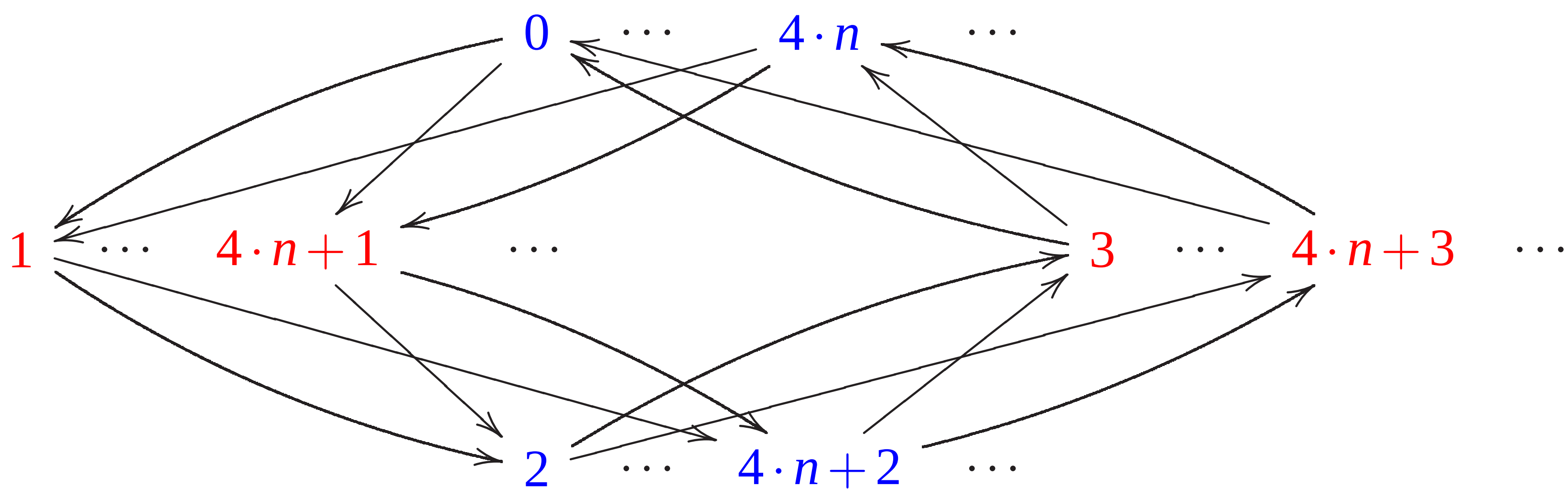
$\therefore$ 

Relation $\to$ *strongly irreflexive, strongly anti-transitive, symmetric.*

♭

Example 3.5 (Partition with 4 infinite blocks)**.** *Set* $\mathbb{N}$*, modulo-4 partition* $\mathrm{P}_4 = \{C_0, C_1, C_2, C_3\}$ *with blocks* $C_0 = \{0, 4, 8, \ldots, 4\cdot n, \ldots\}$, $C_1 = \{1, 5, 9, \ldots, 4\cdot n+1, \ldots\}$, $C_2 = \{2, 6, 10, \ldots, 4\cdot n+2, \ldots\}$ *&* $C_3 = \{3, 7, 11, \ldots, 4\cdot n+3, \ldots\}$ *(cf. Example 2.4, p. 10).*

Relation $\rightarrow$ *on* $\mathbb{N}$*:*



Then: $[4\cdot n] = C_1$, $[4\cdot n+1] = C_2$, $[4\cdot n+2] = C_3$ *&* $[4\cdot n+3] = C_0$.

Thus, relation $\rightarrow$ *induces partition* $\mathrm{P}_3 = \{C_0, C_1, C_2, C_3\}$.

Relation $\rightarrow$ *strong anti-equivalence.* ♭

3.2 Relations for Infinite Partitions

Example 3.6 (Infinite partition with singleton blocks)**.** *Set* $\mathbb{N}$, *partition* $\mathtt{P}_1 = \{\,\{n\} \subseteq \mathbb{N} \,/\, n \in \mathbb{N}\,\}$ *(cf. Example 2.3, p. 8).*

Relation $\rightarrow$ *on* $\mathbb{N}$ *(index* $\mathbb{N}$ *by* $\mathbb{Z}$*):*

$$
\begin{array}{ccccccccccccc}
\cdots & \longrightarrow & 2\cdot n+1 & \longrightarrow & \cdots & \longrightarrow & 3 & \longrightarrow & 1 & & & & \\
 & & & & & & & & \downarrow & & & & \\
 & & & & & & & & 0 & & & & \\
 & & & & & & & & \downarrow & & & & \\
 & & & & & & & & 2 & \longrightarrow & 4 & \longrightarrow & \cdots \longrightarrow 2\cdot n \longrightarrow \cdots
\end{array}
$$

Then: $[2\cdot n+3] = \{2\cdot n+1\}$, $[1] = \{0\}$, $[0] = \{2\}$, $[2\cdot n] = \{2\cdot n+2\}$.
Thus, relation $\rightarrow$ *induces partition* $\mathtt{P}_1 = \{\,\{n\} \subseteq \mathbb{N} \,/\, n \in \mathbb{N}\,\}$.

Relation $\rightarrow$ *strong anti-equivalence.*

Example 3.7 (Infinite finite-block partition)**.** *Set* $\mathbb{Z}$*, partition* $\mathtt{P}_{||} = \{ \mathrm{D}_n \subseteq \mathbb{Z} \,/\, n \in \mathbb{N} \}$*, with* $\mathrm{D}_0 = \{0\}$ *&* $\mathrm{D}_n = \{-n, +n\}$ *(for* $n > 0$*) (cf. Example 2.4: Particular partitions, p. 11).*

Relation $\rightarrow$ *on* $\mathbb{Z}$ *given in Fig. 1 (p. 35).*

Then:

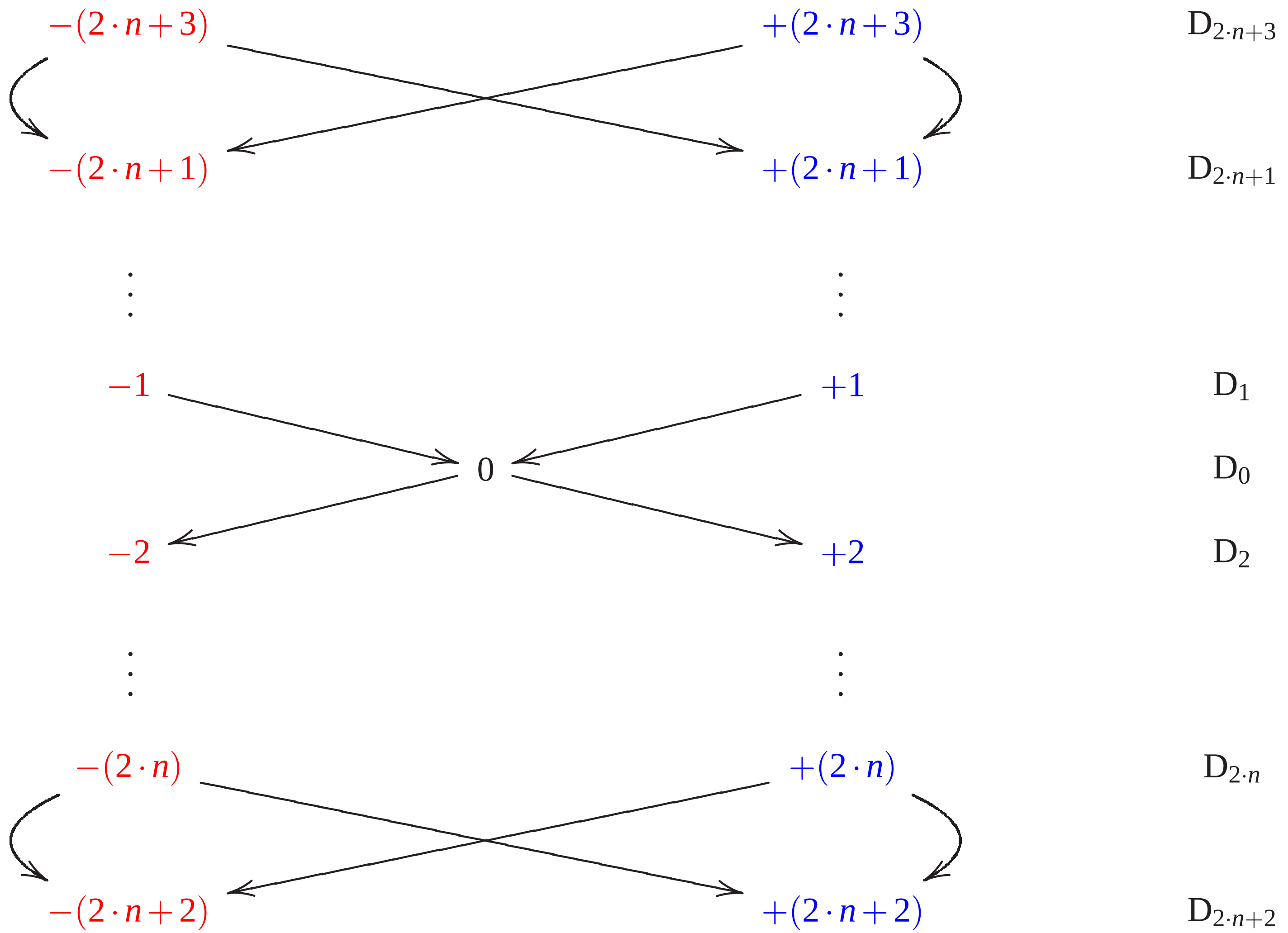
$$\ldots, [-3] = [+3] = \{-1, +1\} = \mathrm{D}_1, [-1] = [+1] = \{0\} = \mathrm{D}_0,$$
$$[0] = \{-2, +2\} = \mathrm{D}_2, [-2] = [+2] = \{-4, +4\} = \mathrm{D}_4,$$
$$[-4] = [+4] = \{-6, +6\} = \mathrm{D}_6, \ldots$$

Thus, relation $\rightarrow$ *induces partition* $\mathtt{P}_{||} = \{ \mathrm{D}_n \subseteq \mathbb{Z} \,/\, n \in \mathbb{N} \}$.

Relation $\rightarrow$ *strong anti-equivalence.*

$\flat$

Figure 1: Relation for absolute-value partition $\mathtt{P}_{||}$ of $\mathbb{Z}$



Example 3.8 (Infinite infinite-block partition). *Set* $\mathbb{N}_+$, *partition* $\mathtt{P}_\infty = \{\, \mathrm{E}_n \subseteq \mathbb{N}_+ \,/\, n \in \mathbb{N} \,\}$, *with* $\mathrm{E}_n = \{\, 2^n \cdot (2 \cdot m + 1) \in \mathbb{N} \,/\, m \in \mathbb{N} \,\}$ *(cf. Example 2.4: Particular partitions, p. 12).*

Relation $\rightarrow$ *on* $\mathbb{N}_+$ *given in Fig 2, p. 37.*

Then:

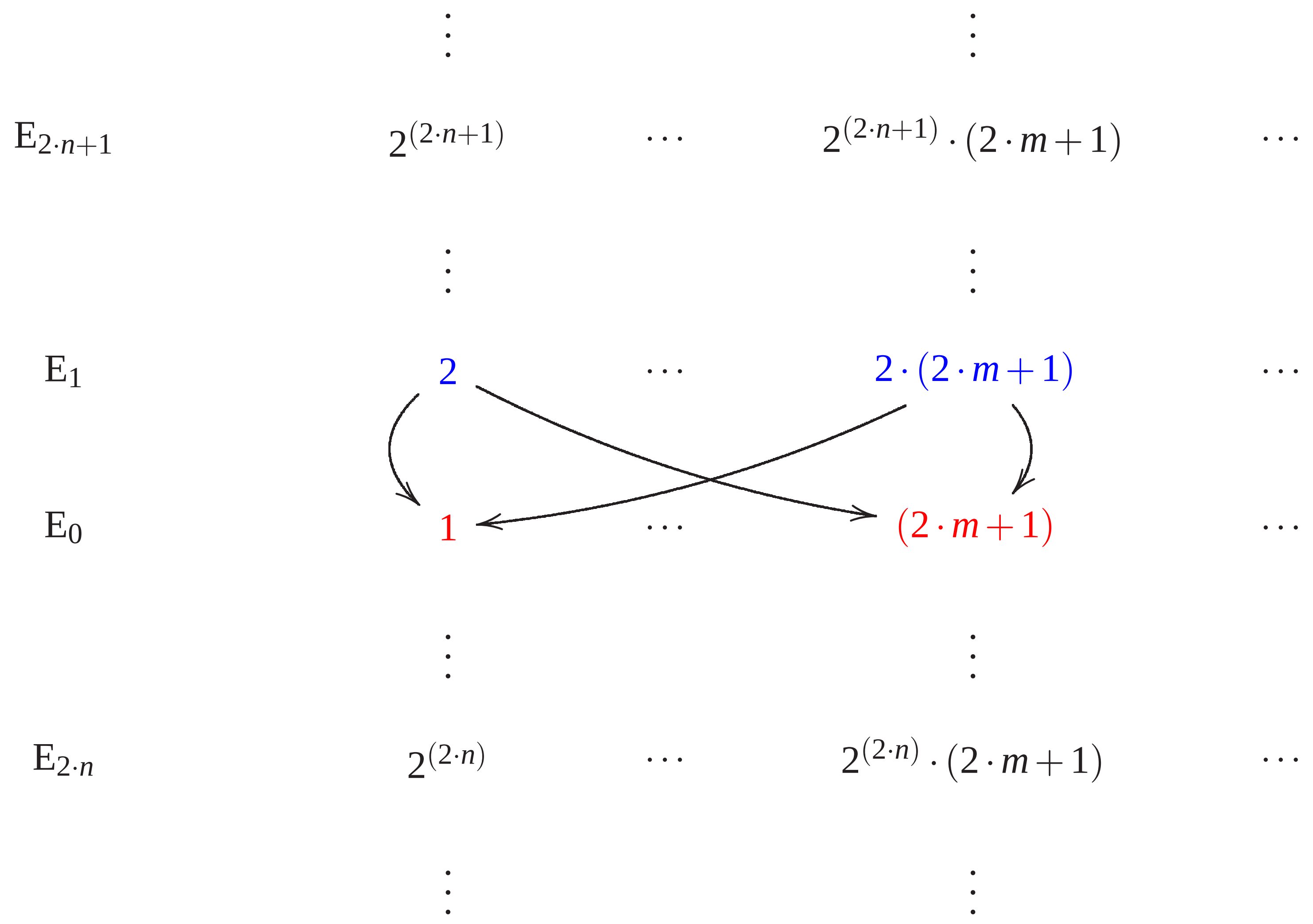
$$\ldots,\ [2^{(2 \cdot n+3)}] = \mathrm{E}_{2 \cdot n+1},\ \ldots,\ [2] = \mathrm{E}_0,\ \ldots,\ [2^{(2 \cdot n)}] = \mathrm{E}_{2 \cdot n+2},\ \ldots$$

Thus, relation $\rightarrow$ *induces partition* $\mathtt{P}_\infty = \{\, \mathrm{E}_n \subseteq \mathbb{N}_+ \,/\, n \in \mathbb{N} \,\}$.

Relation $\rightarrow$ *strong anti-equivalence.*

$\flat$

Figure 2: Relation for infinite-block partition P_∞ of $\mathbb{N}_+$



Outline

4 Relations for Partitions: Analysis

Set $\mathrm{S} \neq \emptyset$, partition $\mathtt{P}$ of S, relation $\longrightarrow$ on S:

$$\longrightarrow \textit{ induces } \mathtt{P} \quad (\longrightarrow \lhd \mathtt{P}) \qquad \text{iff} \qquad \mathrm{S}/\!\longrightarrow \; = \mathtt{P}.$$

∂

4.1 Limitative Results

Proposition 4.1 (Relation for one-block partition). *Set* $\mathrm{S} \neq \emptyset$, *one-block partition* $\mathtt{Q} = \{\mathrm{S}\}$, *relation* $\longrightarrow$ *on* S.

$$\longrightarrow \lhd \mathtt{Q} \quad \Rightarrow \quad \longrightarrow \; = \underbrace{\mathrm{S} \times \mathrm{S}}_{\substack{\textit{full} \\ \textit{equivalence}}} .$$

$\natural$

(Cf. Example 3.3: Single finite block, p. 30).

Proposition 4.2 (Relation for two-block partition)**.** *Set* S $\neq \emptyset$, *2-bock partition* $\mathtt{P} = \{B, C\}$, *relation* $\rightarrow$ *on* S.

$$\left(\begin{array}{c} \rightarrow \ \lhd \ \mathtt{P} \\ \underbrace{\rightarrow\!: \ \mathsf{siR}}_{\substack{\textit{strongly} \\ \textit{irreflexive}}} \end{array} \right) \quad \Rightarrow \quad \rightarrow \ = \ \underbrace{(B \times C) \cup (C \times B)}_{\mathsf{saT},\ \mathtt{Smm}}.$$

♮

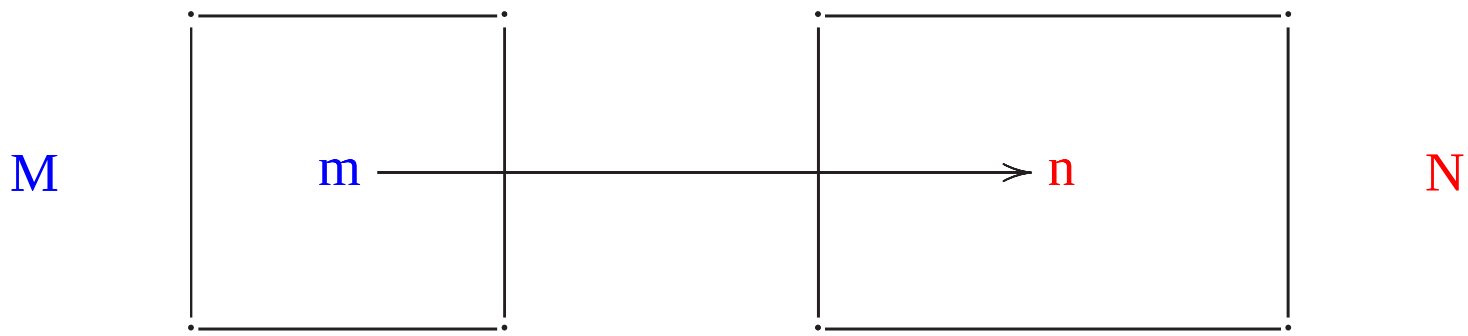
(Cf. Example 3.4: Two-singleton partition, p. 31).

4.2 Relation from Partition

Set S, subsets $M, N \subseteq S$, relation $\rightarrow$ *matches* M *to* N iff

$$\forall a, b \in S: \quad a \rightarrow b \quad \Leftrightarrow \quad \Big(a \in M \quad \& \quad b \in N \Big)$$

∂



Lemma 4.1 (Matching relation)**.** *Set* S*, subsets* $M, N \subseteq S$*, relation* $\rightarrow$ *matching* M *to* N*. Given element* $m \in M$*:*

1. $\forall b \in S: \quad m \rightarrow b \quad \Leftrightarrow \quad b \in N$;
2. *Class:* $[m] = N$.

♮

$$\begin{pmatrix} \text{Partition } \mathtt{P} \text{ of } \mathrm{S} \\ \text{transformation } \mathtt{t} : \mathtt{P} \longrightarrow \mathtt{P} \end{pmatrix} \quad \mapsto \quad \textit{relation of } \mathtt{t}:$$

$$\mathrm{a} \xrightarrow{\mathtt{t}} \mathrm{b} \quad \Leftrightarrow \quad \mathtt{P}(\mathrm{a})^{\mathtt{t}} = \mathtt{P}(\mathrm{b})$$

∂

Notation $\mathtt{Fx}(\mathtt{f})$: *fix-point set of* $\mathtt{f}$ ∂

Proposition 4.3 (Relation of partition transformation)**.** *Partition* $\mathtt{P}$ *of* S, *function* $\mathtt{t} : \mathtt{P} \longrightarrow \mathtt{P}$.

1. $\mathtt{Fx}(\mathtt{t}) = \emptyset \quad \Rightarrow \quad \xrightarrow{\mathtt{t}}$: siR, saT *(str. irreflexive, anti-transitive).*
2. $\mathtt{Fx}(\mathtt{t}^2) = \emptyset \quad \Rightarrow \quad \xrightarrow{\mathtt{t}}$: saS *(strongly asymmetric).*
3. $\forall \mathrm{B} \in \mathtt{P}$: $\left(\xrightarrow{\mathtt{t}} \textit{matches } \mathrm{B} \textit{ to } \mathrm{B}^{\mathtt{t}} \quad \therefore \quad \forall \mathrm{a} \in \mathrm{B} : [\mathrm{a}] = \mathrm{B}^{\mathtt{t}} \right)$.
4. $\mathtt{t}$: *bijective* $\quad \Rightarrow \quad \xrightarrow{\mathtt{t}} \lhd \mathtt{P}$.

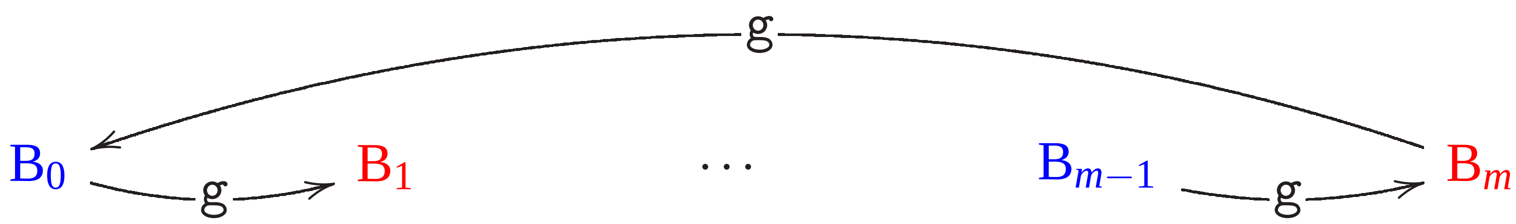
♮

Outline

5 Relations for Non-trivial Partitions

Permutations of partition $\mathtt{P} = \{B_{\mathtt{i}} \subseteq \mathtt{S} / \mathtt{i} \in \mathtt{I}\}$

Lemma 5.1 (Finite partitions). $\mathtt{I} = \{0, \ldots, m\}$ *($m \in \mathbb{N}$).*
Partition $\mathtt{P} = \{B_0, \ldots, B_m\}$ *(cf. Examples 3.2, p. 29, and 3.5, p. 32):*

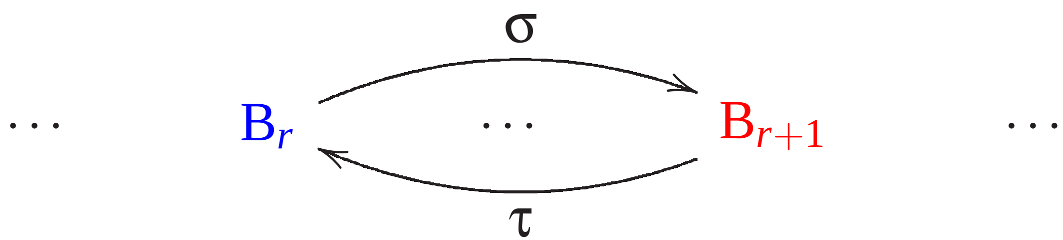


$m \backslash$ *fixpoint*	$\mathrm{Fx}(g)$	$\mathrm{Fx}(g^2)$
$m = 0$	$\{B_0\}$	$\{B_0\}$
$m = 1$	$\emptyset$	$\{B_0\}$
$m > 1$	$\emptyset$	$\emptyset$

Lemma 5.2 (Infinite partitions). *Concrete index set*

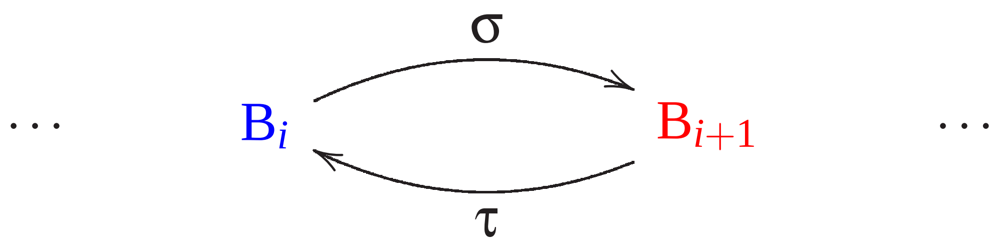
1. *Permutations of* $\mathtt{P} = \{\mathrm{B}_r \subseteq \mathrm{S} \,/\, r \in \mathbb{R}\}$ *successor & predecessor*

$$\sigma(\mathrm{B}_r) \;\coloneqq\; \mathrm{B}_{r+1} \qquad\qquad \tau(\mathrm{B}_r) \;\coloneqq\; \mathrm{B}_{r-1}$$

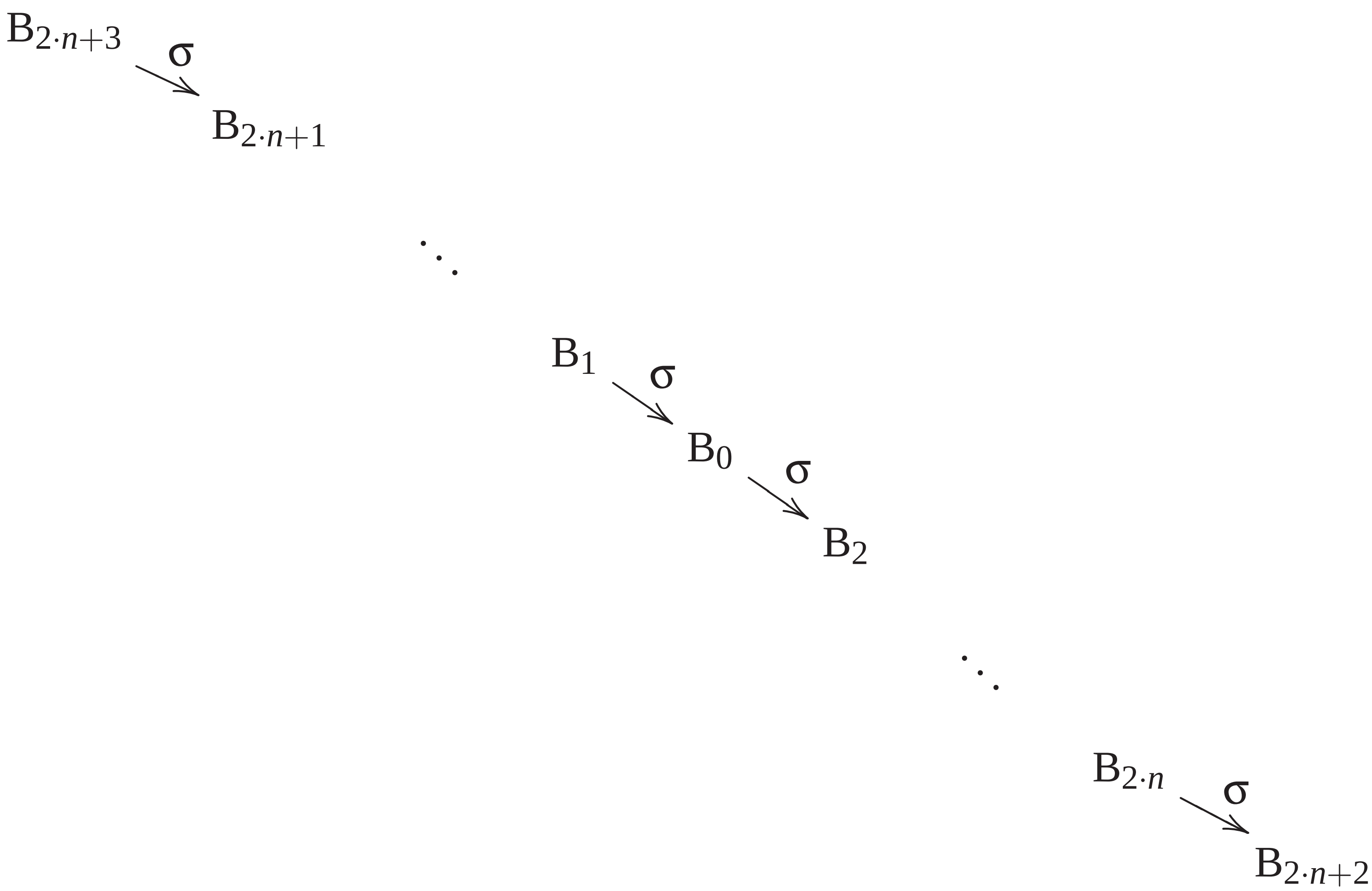


Fixpoints $\qquad \mathtt{Fx}(\sigma) \;=\; \emptyset \;=\; \mathtt{Fx}(\sigma^2)$

2. *Integers:* $\mathtt{I} = \mathbb{Z}$ *restriction*



3. Naturals: $\mathtt{I} = \mathbb{N}$ *bijection* $\mathbb{N} \to \mathbb{Z}$ *(cf. Example 3.6, p. 33):*



Infinite Partitions (general case)

Proposition 5.1 (Permutation for infinite partition)**.**

$$\left(\quad \textit{Partition}\ \mathtt{P} \qquad \textit{infinite}\ \mathtt{P} \quad \right)$$

$$\Downarrow$$

$$\left(\quad \exists\ \textit{permutation}\ \mathtt{t} : \mathtt{P} \to \mathtt{P} \qquad \mathtt{Fx}(\mathtt{t}) = \mathtt{Fx}(\mathtt{t}^2) = \emptyset \quad \right)$$

♮

Compactness & downward Löwenheim-Skolem,

see van Dalen pp. 121, 123 (Exercise 10 (v))**.**

Theorem 5.1 (Relation for non-trivial partition). *Partition* $\mathtt{P}$ *of* $\mathrm{S} \neq \emptyset$ *with* $|\,\mathtt{P}\,| \geq 2$*, there is a non-equivalence* (siR, saT) $\xrightarrow{\mathtt{t}}$ *inducing* $\mathtt{P}$*:*

$$(=)\quad |\,\mathtt{P}\,| = 2 \quad\Rightarrow\quad \xrightarrow{\mathtt{t}}\text{: } \mathtt{Smm} \qquad \textit{(symmetric);}$$

$$(>)\quad |\,\mathtt{P}\,| > 2 \quad\Rightarrow\quad \xrightarrow{\mathtt{t}}\text{: } \mathsf{saS} \qquad \textit{(strong anti-equivalence).}$$

♮

$\|\,\mathtt{P}\,\| \geq 2$	Rfl?	Smm?	Trn?
$\|\,\mathtt{P}\,\| = 2$	siR	Smm	saT
$\|\,\mathtt{P}\,\| > 2$	siR	saS	saT

Outline

6 Relations Inducing Partitions

Theorem 6.1 (Relation inducing partition). *Partition* $\mathtt{P}$ *can be induced by a non-equivalence iff* $|\,\mathtt{P}\,| > 1$.

♮

Theorem 6.2 (Summary). *Relation inducing* $\mathtt{P} = \{\, B_{\mathtt{i}} \subseteq S \,/\, \mathtt{i} \in \mathtt{I} \,\}$

Partition $\mathtt{P}$	Rfl?	Smm?	Trn?	*Exhibit?*
Infinite $\mathtt{I}$	siR	saS	saT	*No*
Infinite $\mathtt{I} \in \{\mathbb{R}, \mathbb{Z}, \mathbb{N}\}$	siR	saS	saT	*Yes*
Finite $|\,\mathtt{I}\,| > 2$	siR	saS	saT	*Yes*
Finite $|\,\mathtt{I}\,| = 2$	siR	Smm	saT	*Yes*
Finite $|\,\mathtt{I}\,| = 1$	Rfl	Smm	Trn	*Yes*

♮

Remark 6.1 (Examples). *Relations for partition* $\mathtt{P} = \{\mathtt{B_i} \subseteq \mathtt{S} \,/\, \mathtt{i} \in \mathtt{I}\}$

1. *Infinite* $\mathtt{I} \in \{\mathbb{R}, \mathbb{Z}, \mathbb{N}\}$ (siR, saS, saT) saE:

   $\mathtt{P_1}$ *in 3.6 (Infinite partition with singleton blocks, p. 33);*

   $\mathtt{P_{||}}$ *in 3.7 (Infinite finite-block partition, p. 34);*

   $\mathtt{P_\infty}$ *in 3.8 (Infinite infinite-block partition, p. 36).*

2. *Finite* $|\mathtt{I}| > 2$ (siR, saS, saT) saE:

   $\mathtt{P_3}$ *in 3.2 (Partition with 3 finite blocks, p. 29);*

   $\mathtt{P_4}$ *in 3.5 (Partition with 4 infinite blocks, p. 32).*

3. *Finite* $|\mathtt{I}| = 2$ (siR, Smm, saT) $\mathtt{E}\not{\mathtt{q}}\mathtt{v}$:

   $\mathtt{P}$ *in 3.4 (Two-singleton partition, p. 31).*

4. *Finite* $|\mathtt{I}| = 1$ (Rfl, Smm, Trn) Eqv:

   $\mathtt{P_0}$ *in 3.3 (Single finite block, p. 30).*

√

Other Aspects

1. When does a relation induce a given partition?

   Similar to Proposition 2.1: Quotient and partition, p. 20.

2. When is a relation "uniform" w. r. t. a partition?

   When one has: same classes iff same blocks

   (cf. Example 3.2: Partition with 3 finite blocks, p. 29).

3. Can one have non-uniform relations inducing a partition?

   Yes; see Example 3.1: Partition with 2 finite blocks (first relation), p. 27.

4. Do we need partition permutations?

   Yes, if we wish uniform relations inducing the partition.

Retrospect

A Details

A.1 Details on Partitions and Relations (Sct. 2)

Remark 2.2 (Partition block). *Unique block with element.*
Partition $\mathtt{P}$ *of* S*, element* $\mathrm{s} \in \mathrm{S}$*:* $\exists!$ *block* $\mathrm{B} \in \mathtt{P}$*, s. t.* $\mathrm{s} \in \mathrm{B}$. $\surd$

[$(\bigcup)$ $\Rightarrow$ $\exists$ $(\cap)$ $\Rightarrow$ $!$ (cf. p. 6)]

Propostion 2.1 (Quotient and partition) *Given relation* $\rightarrow$ *on set* $S \neq \emptyset$, *quotient* $S/\!\rightarrow$ *is a partition iff* $\rightarrow$ *satisfies the 3 conditions:*

(δ) $S \subseteq \mathrm{Dmn}(\rightarrow)$ $\qquad \forall b \in S\ \exists a \in S:\ a \rightarrow b$

(ι) $S \subseteq \mathrm{Img}(\rightarrow)$ $\qquad \forall b \in S\ \exists a \in S:\ a \rightarrow b$

$$(\gamma) \quad \exists s \in S \left(\begin{array}{ccc} b & & \\ & \searrow & \\ & & s \\ & \nearrow & \\ c & & \end{array} \right) \quad \Rightarrow \quad \forall t \in S \left(\begin{array}{ccc} b & \longrightarrow & t \\ & \Downarrow & \\ c & \longrightarrow & t \end{array} \right)$$

$\natural$

Proof.

$(\delta) \Leftrightarrow (\emptyset)$, $(\iota) \Leftrightarrow (\bigcup)$, $(\gamma) \Leftrightarrow (\bigcap)$.

$\dashv$

Reflexion points & confluence property (γ) cf. pp. 20, 55

Lemma 2.1 (Reflexive confluence) *Relation $\rightarrow$ with confluence (γ)*

1. b*: reflexion point* $\Rightarrow$ (a, b, c)*: transitive triple*

2. a, b*: reflexion points* $\Rightarrow$ (a, b)*: symmetric pair*

3. $\rightarrow$ *reflexive* $\Rightarrow$ $\left(\begin{array}{l} \rightarrow \textit{symmetric} \\ \rightarrow \textit{transitive} \end{array} \right)$ $\therefore$ *equivalence*

$\natural$

Proof.

1.
$$a \longrightarrow b \quad \Rightarrow \quad \begin{array}{ccc} b & & \\ & \searrow & \\ & & b \\ & \nearrow & \\ a & & \end{array} \quad \therefore \quad \begin{array}{c} b \longrightarrow c \\ \Downarrow \\ a \longrightarrow c \end{array}$$

2.
$$a \longrightarrow b \quad \Rightarrow \quad \begin{array}{ccc} a & & \\ & \searrow & \\ & & b \\ & \nearrow & \\ b & & \end{array} \quad \therefore \quad \begin{array}{c} a \longrightarrow a \\ \Downarrow \\ b \longrightarrow a \end{array}$$

⊣

Corollary 2.1 (Partition from non-equivalence). *Relation* $\longrightarrow$ *on* $\mathrm{S} \neq \emptyset$

$$\left(\begin{array}{l} \mathrm{S}/\!\longrightarrow : \ \textit{partition} \\ \longrightarrow \ \textit{non-equivalence} \end{array} \right) \quad \Rightarrow \quad \longrightarrow: \textit{non-reflexive.}$$

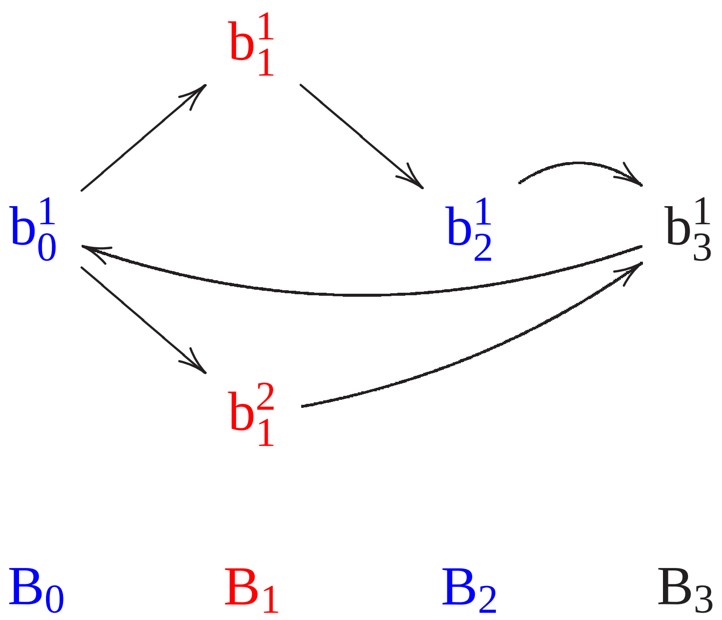
♮

Proof. Lemma 2.1: Reflexive confluence, pp. 22, 56. ⊣

A.2 Details on Relations for Partitions: Examples (Sct. 3)

Strong anti-equivalences for partitions non-uniform

Example A.1 (Four-block partition, 5 elements)**.**



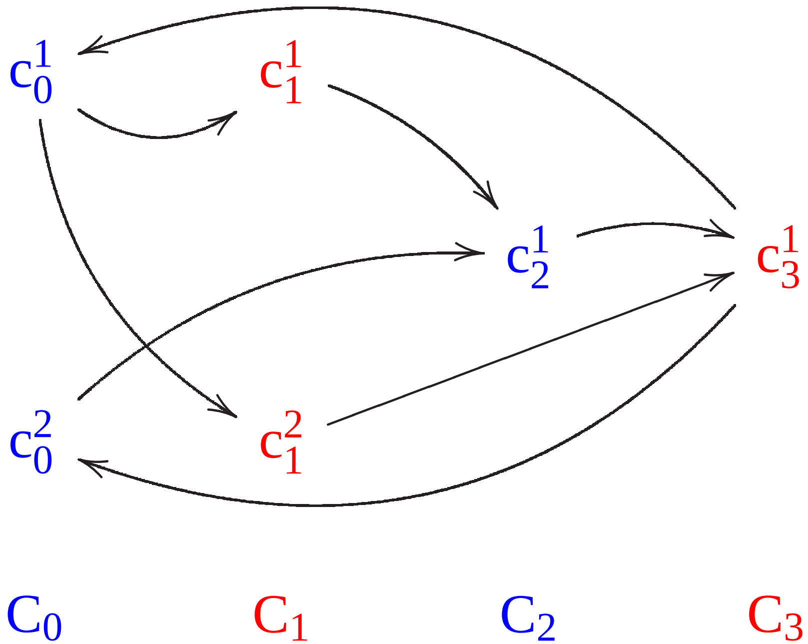
$$[b_1^1] = B_2$$

$$[b_0^1] = B_1 \quad [b_2^1] = B_3 \quad [b_3^1] = B_0$$

$$[b_1^2] = B_3$$

♭

Example A.2 (Four-block partition, 6 elements)**.**



$[c_0^1] = C_1 \quad [c_1^1] = C_2$

$[c_2^1] = C_3 \quad [c_3^1] = C_0$

$[c_0^2] = C_2 \quad [c_1^2] = C_3$

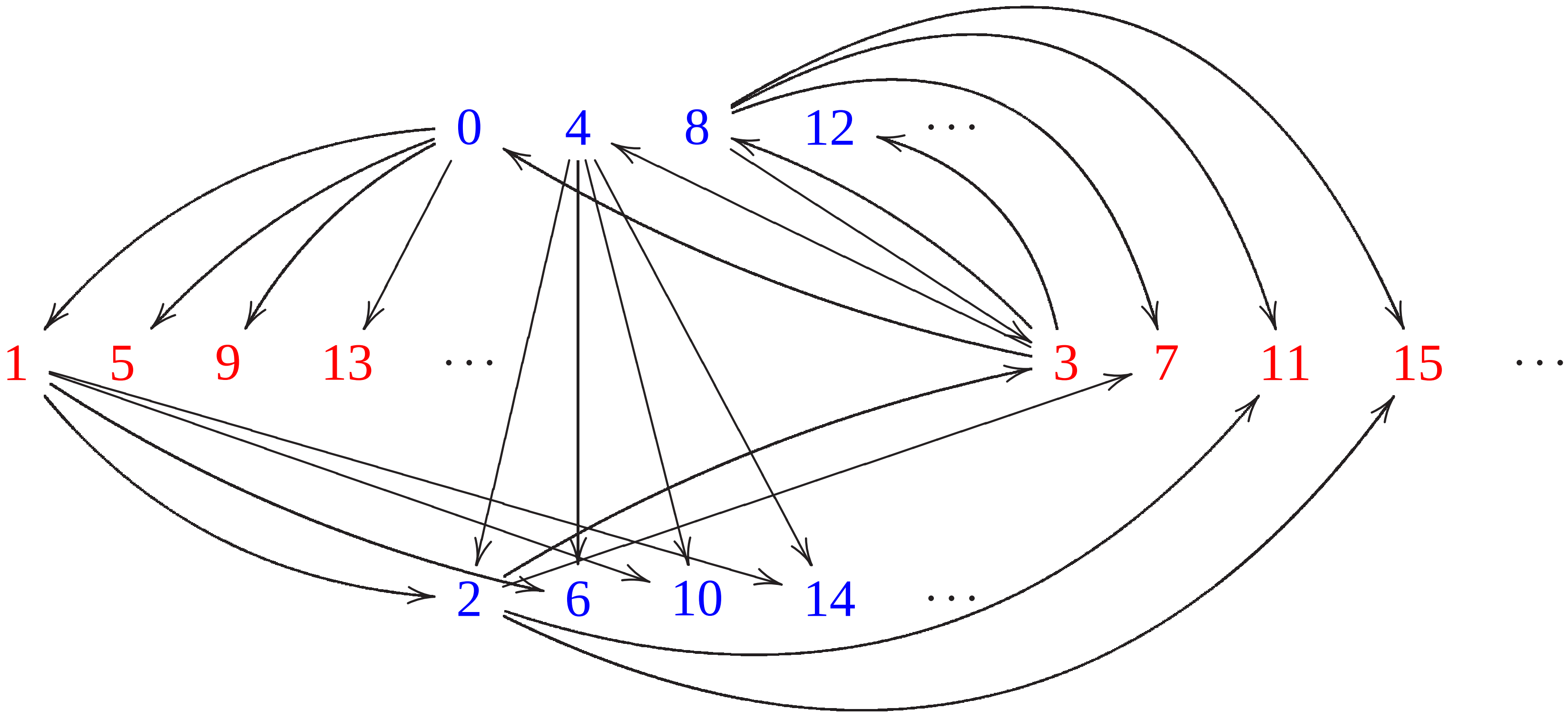
♭

Example A.3 (Infinite partition, finite blocks). *Set* $\mathbb{N}$, *modulo-4 partition* $\mathrm{P}_4 = \{C_0, C_1, C_2, C_3\}$ *with blocks* $C_0 = \{0, 4, 8, \ldots, 4 \cdot n, \ldots\}$, $C_1 = \{1, 5, 9, \ldots, 4 \cdot n + 1, \ldots\}$, $C_2 = \{2, 6, 10, \ldots, 4 \cdot n + 2, \ldots\}$ *&* $C_3 = \{3, 7, 11, \ldots, 4 \cdot n + 3, \ldots\}$ *(cf. Example 2.4, p. 10).*

Desired classes:

$$
\begin{array}{lllll}
[0] = C_1 & [4] = C_2 & [8] = C_3 & [12] = C_1 & \ldots \\
[1] = C_2 & [5] = C_3 & [9] = C_0 & [13] = C_2 & \ldots \\
[2] = C_3 & [6] = C_0 & [10] = C_1 & [14] = C_3 & \ldots \\
[3] = C_0 & [7] = C_1 & [11] = C_2 & [15] = C_0 & \ldots
\end{array}
$$

Relation $\rightarrow$ *on* $\mathbb{N}$:



Example A.4 (Infinite partition, finite blocks). *Set* $\mathbb{Z}$*, partition* $\mathsf{P}_{||} = \{\mathrm{D}_n \subseteq \mathbb{Z} \,/\, n \in \mathbb{N}\}$*, with* $\mathrm{D}_0 = \{0\}$ *&* $\mathrm{D}_n = \{-n, +n\}$ *(for* $n > 0$*)* *(cf. Example 2.4: Particular partitions, p. 11).*

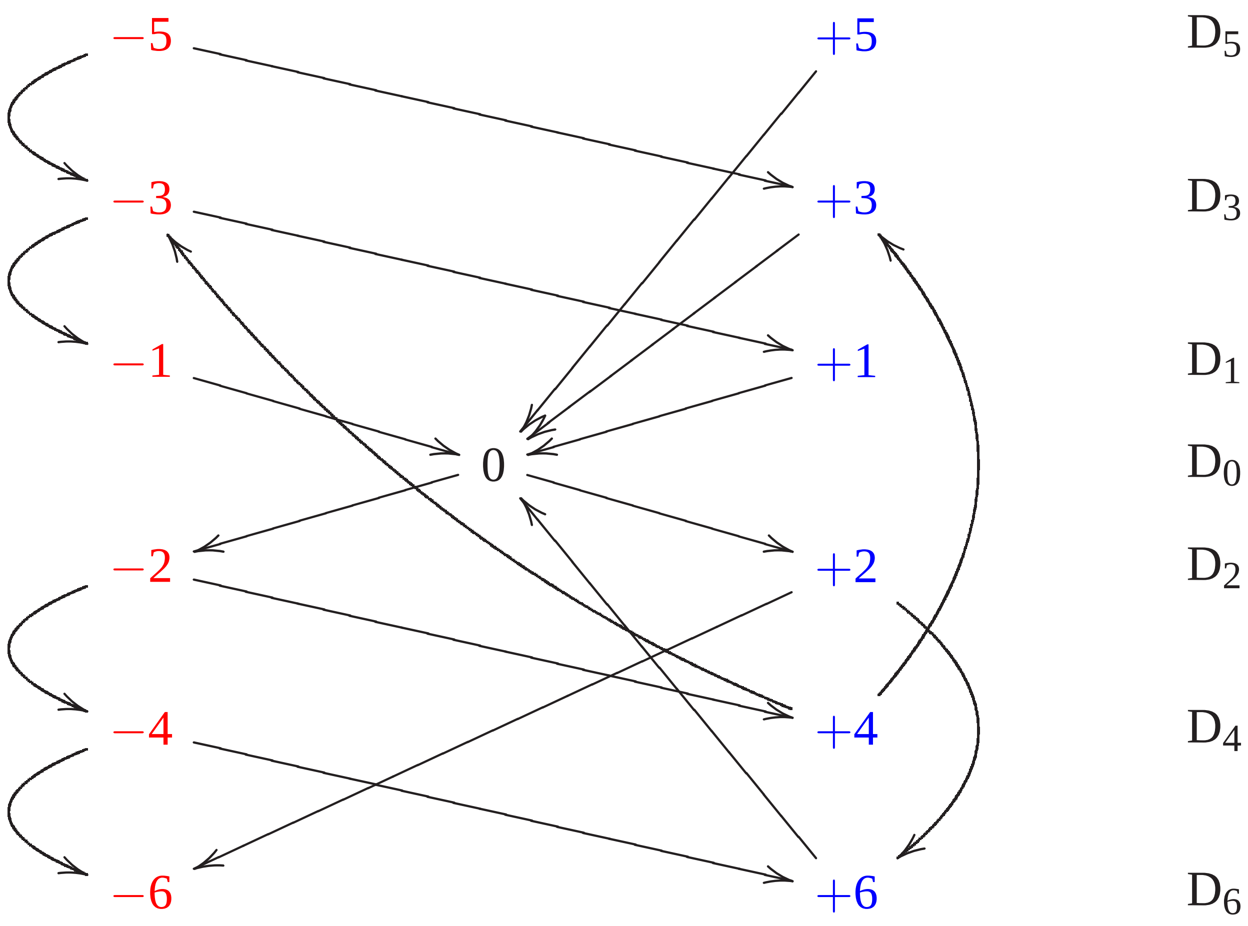
Desired classes:

$$\ldots \quad [-5] = \mathrm{D}_1 \quad [-3] = \mathrm{D}_1 \quad [-1] = \mathrm{D}_0$$

$$[0] = \mathrm{D}_2 \qquad\qquad [-2] = \mathrm{D}_4$$

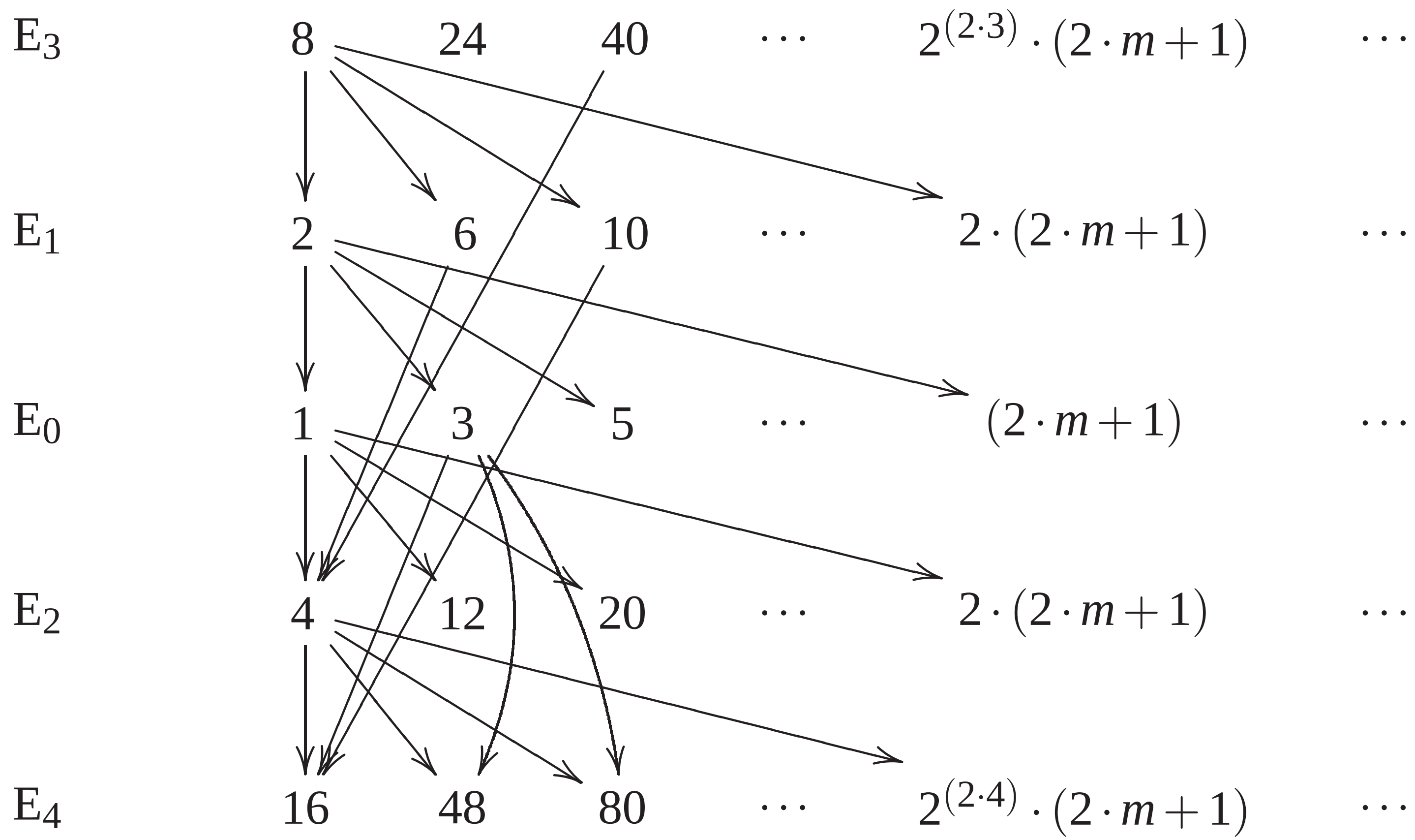
$$[+2] = \mathrm{D}_6 \qquad\qquad [+1] = [+4] = [+6] = \ldots = \mathrm{D}_0$$

Relation $\rightarrow$ *on* $\mathbb{Z}$*:*



Example A.5 (Infinite partition, infinite blocks). *Set* $\mathbb{N}_+$, *partition* $\mathtt{P}_\infty = \{\, \mathrm{E}_n \subseteq \mathbb{N}_+ \,/\, n \in \mathbb{N} \,\}$, *with* $\mathrm{E}_n = \{\, 2^n \cdot (2 \cdot m + 1) \in \mathbb{N} \,/\, m \in \mathbb{N} \,\}$ *(cf. Example 2.4: Particular partitions, p. 12).*

Relation $\rightarrow$ *on* $\mathbb{N}_+$*:*



A.3 Details on Relations for Partitions: Analysis (Sct. 4)

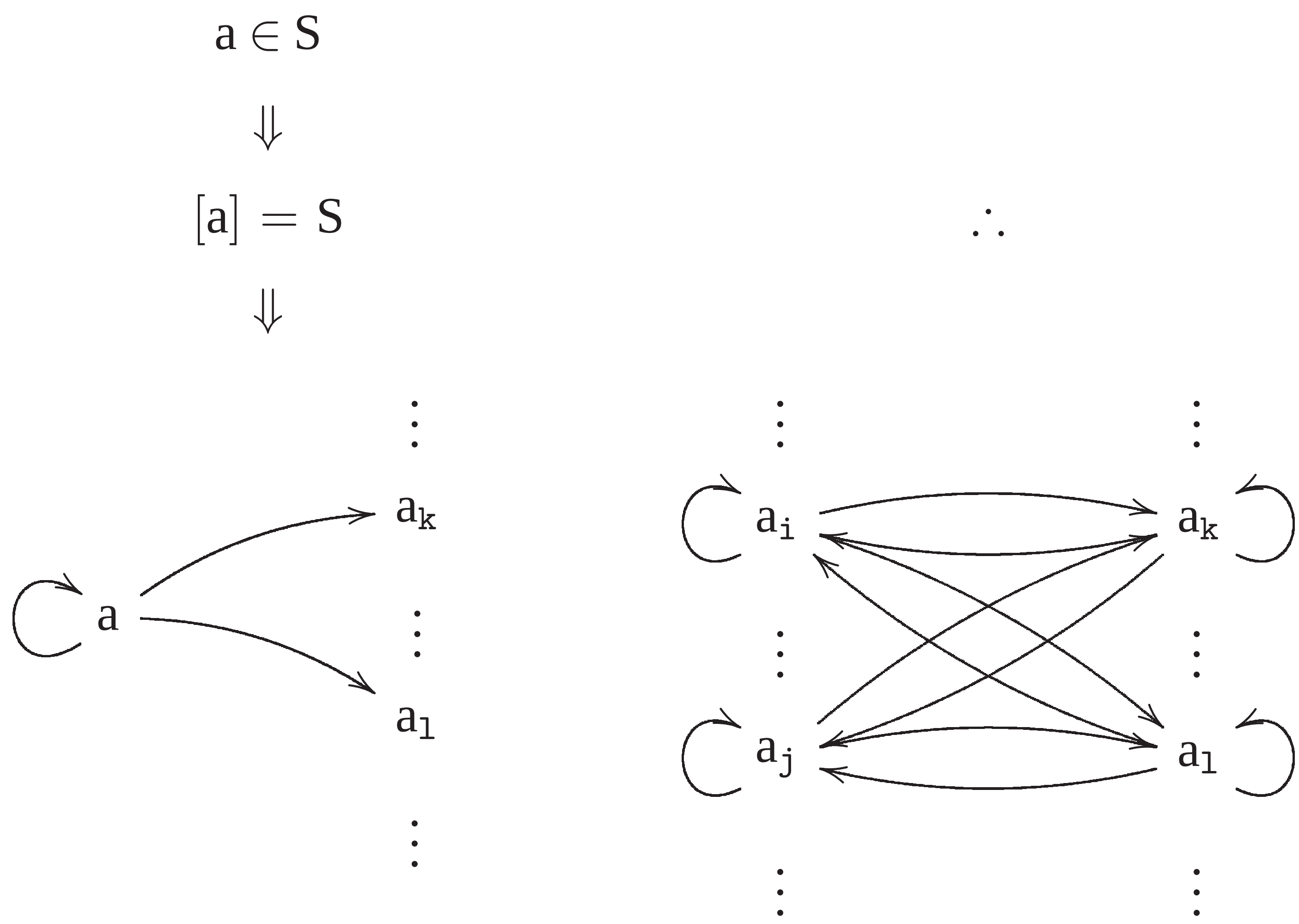
Proposition 4.1 (Relation for one-block partition). *Set* S $\neq \emptyset$, *one-block partition* $\mathtt{Q} = \{\mathrm{S}\}$, *relation* $\rightarrow$ *on* S.

$$\rightarrow \;\lhd\; \mathtt{Q} \quad \Rightarrow \quad \rightarrow \;=\; \underbrace{\mathrm{S} \times \mathrm{S}}_{\substack{\textit{full}\\ \textit{equivalence}}} \quad . \qquad \natural$$

Proof.

See Fig. 3, p. 66 (cf. Example 3.3: Single finite block, p. 30). $\dashv$

Figure 3: Relation → inducing one-block partition $\mathtt{Q} = \{S\}$



Proposition 4.2 (Relation for two-block partition). *Set* S $\neq \emptyset$, *2-bock partition* $\mathtt{P} = \{\mathrm{B}, \mathrm{C}\}$, *relation* $\rightarrow$ *on* S.

$$\left(\begin{array}{c} \rightarrow \;\lhd\; \mathtt{P} \\ \underbrace{\rightarrow\!: \;\mathsf{siR}}_{\substack{\textit{strongly} \\ \textit{irreflexive}}} \end{array} \right) \quad \Rightarrow \quad \rightarrow \;=\; \underbrace{(\mathrm{B} \times \mathrm{C}) \cup (\mathrm{C} \times \mathrm{B})}_{\mathtt{saT},\ \mathtt{Smm}}.$$

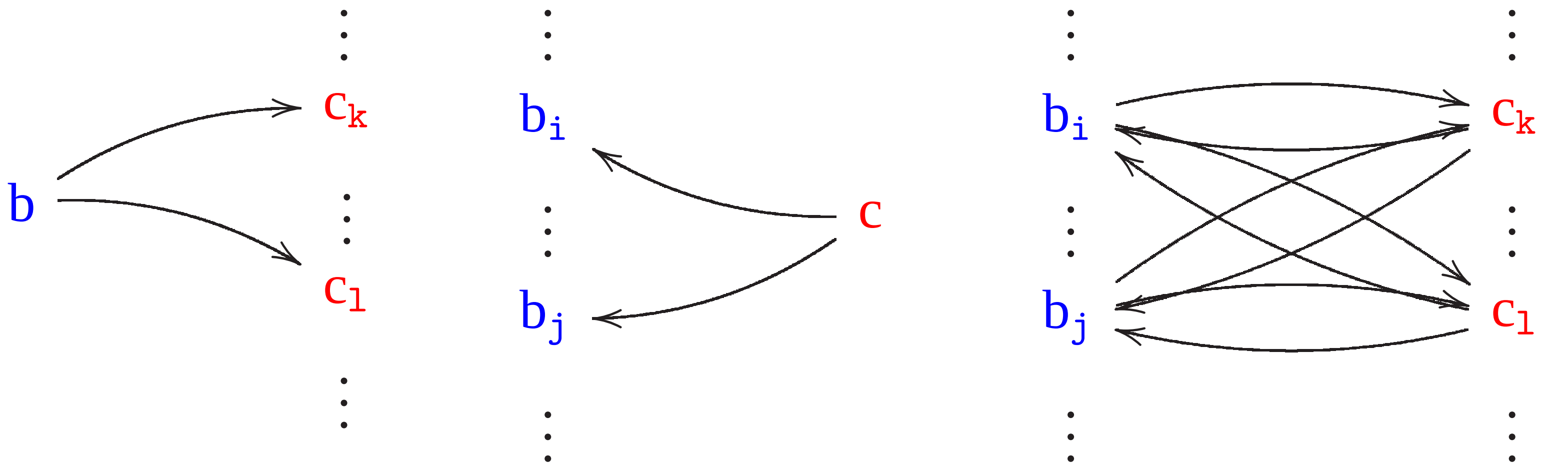
♮

Proof.

See Fig. 4, p. 68 (cf. Example 3.4: Two-singleton partition, p. 31). ⊣

Figure 4: Strongly irreflexive relation $\rightarrow$ inducing $\mathtt{P} = \{B, C\}$

$b \in B$	$c \in C$	
$\Downarrow$	$\Downarrow$	
$[b] \neq B$	$[c] \neq C$	
$\Downarrow$	$\Downarrow$	
$[b] = C$	$[c] = B$	$\therefore$
$\Downarrow$	$\Downarrow$	



Lemma 4.1 (Matching relation) Set S, subsets $M, N \subseteq S$, relation $\rightarrow$ matching M to N. Given element $m \in M$:

1. $\forall b \in S: \quad m \rightarrow b \quad \Leftrightarrow \quad b \in N$;

2. Class: $[m] = N$.

$\natural$

Proof.

1. $m \in M \quad \Rightarrow \quad \left(m \rightarrow b \quad \Leftrightarrow \quad b \in N \right)$.

2. Follows from 1. (as $b \in [m] \quad \Leftrightarrow \quad m \rightarrow b$).

$\dashv$

Proposition 4.3 (Relation of partition transformation). *Partition* $\mathtt{P}$ *of* S, *function* $\mathtt{t} : \mathtt{P} \longrightarrow \mathtt{P}$.

1. $\mathtt{Fx}(\mathtt{t}) = \emptyset \qquad \Rightarrow \qquad \xrightarrow{\mathtt{t}}$: siR, saT *(str. irreflexive, anti-transitive)*.

2. $\mathtt{Fx}(\mathtt{t}^2) = \emptyset \qquad \Rightarrow \qquad \xrightarrow{\mathtt{t}}$: saS *(strongly asymmetric)*.

3. $\forall \mathrm{B} \in \mathtt{P}$: $\Big(\xrightarrow{\mathtt{t}}$ *matches* B *to* $\mathrm{B}^{\mathtt{t}} \quad \therefore \quad \forall \mathrm{a} \in \mathrm{B} : [\mathrm{a}] = \mathrm{B}^{\mathtt{t}} \Big)$.

4. $\mathtt{t}$: *bijective* $\qquad \Rightarrow \qquad \xrightarrow{\mathtt{t}} \lhd \mathtt{P}$.

♮

Proof.

1. $\mathtt{Fx}(\mathtt{t}) = \emptyset$:

(siR) $s \xrightarrow{\mathtt{t}} s \quad \Rightarrow \quad \mathtt{P}(s)^{\mathtt{t}} = \mathtt{P}(s) \quad \therefore \quad s \in \mathtt{Fx}(\mathtt{t})$;

(saT) $$\left.\begin{array}{lll} a \xrightarrow{\mathtt{t}} b & \Rightarrow & \mathtt{P}(a)^{\mathtt{t}} = \mathtt{P}(b) \\ a \xrightarrow{\mathtt{t}} c & \Rightarrow & \mathtt{P}(a)^{\mathtt{t}} = \mathtt{P}(c) \end{array}\right\} \Rightarrow \quad \mathtt{P}(b) = \mathtt{P}(c)$$

$\therefore$

$$b \xrightarrow{\mathtt{t}} c \quad \Rightarrow \quad \mathtt{P}(b)^{\mathtt{t}} = \mathtt{P}(c) = \mathtt{P}(b) \quad \therefore \quad b \in \mathtt{Fx}(\mathtt{t}).$$

2. $$\left.\begin{array}{lll} a \xrightarrow{\mathtt{t}} b & \Rightarrow & \mathtt{P}(a)^{\mathtt{t}} = \mathtt{P}(b) \\ b \xrightarrow{\mathtt{t}} a & \Rightarrow & \mathtt{P}(b)^{\mathtt{t}} = \mathtt{P}(a) \end{array}\right\} \Rightarrow \quad \mathtt{P}(a)^{\mathtt{t}^2} = \mathtt{P}(b)^{\mathtt{t}} = \mathtt{P}(a)$$

$\therefore \quad a \in \mathtt{Fx}(\mathtt{t}^2)$.

3. $a \xrightarrow{\mathtt{t}} b \quad \Leftrightarrow \quad \mathtt{P}(a)^{\mathtt{t}} = \mathtt{P}(b) \quad \Leftrightarrow \quad a \in \mathtt{P}(a) \;\&\; b \in \mathtt{P}(a)^{\mathtt{t}}$.

4. $\forall s \in S : [s] = \mathtt{P}(s)^{\mathtt{t}}$ $\qquad \forall B \in \mathtt{P} \, \exists t \in B^{\mathtt{t}^{-1}} : [t] = B.$

⊣

A.4 Details on Relations for Non-trivial Partitions (Sct. 5)

Proposition 5.1 (Permutation for infinite partition).

$$\left(\textit{Partition}\ \mathtt{P} \qquad \textit{infinite}\ \mathtt{P} \right)$$

$$\Downarrow$$

$$\left(\exists\ \textit{permutation}\ \mathtt{t} : \mathtt{P} \to \mathtt{P} \qquad \mathtt{Fx}(\mathtt{t}) = \mathtt{Fx}(\mathtt{t}^2) = \emptyset \right)$$

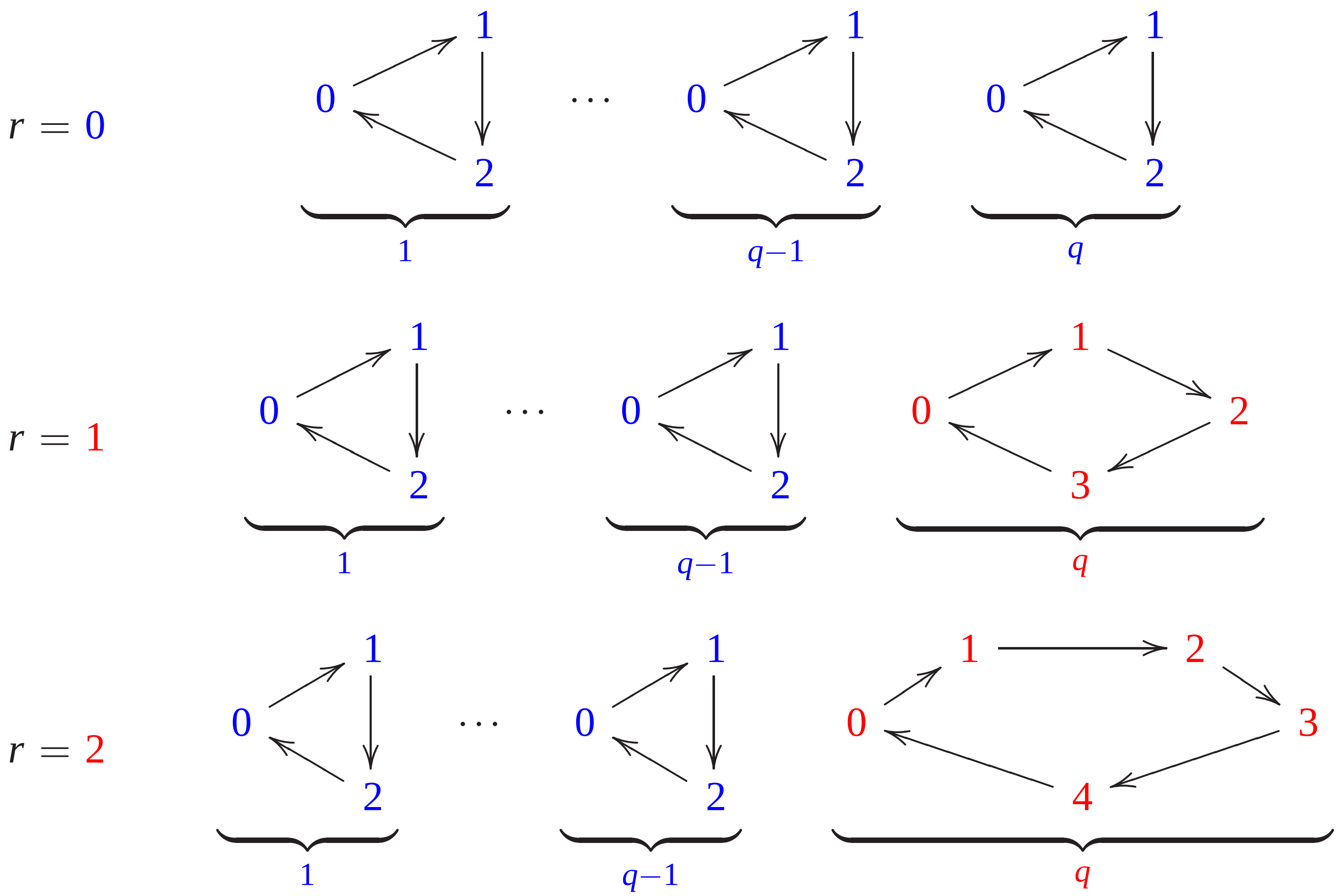
♮

Proof. Compactness & downward Löwenheim-Skolem,
see van Dalen pp. 121, 123 (Exercise 10 (v)).

(First-order theory Γ: f bijective, $\mathtt{Fx}(f) = \mathtt{Fx}(f^2) = \emptyset$,
$\{\neg \mathsf{c} \doteq \mathsf{d} \,/\, \mathsf{c} \neq \mathsf{d} \in \mathsf{C}\}$, set C of new constants c_{B}, for each block $\mathrm{B} \in \mathtt{P}$.
Γ finitely consistent (Lemma 5.1: Finite Partitions, p. 44).
Γ has model $\mathfrak{M}$ with $|\,\mathfrak{M}\,| = |\,\mathtt{P}\,|$. Use bijection to transfer $f^{\mathfrak{M}}$ to $\mathtt{P}$.)

⊣

Remark A.1 (Permutations of n elements). $n = 3 \cdot q + r, \quad 0 \le r < 3$



$\surd$

Theorem 5.1 (Relation for non-trivial partition). *Partition* $\mathtt{P}$ *of* $\mathrm{S} \neq \emptyset$ *with* $|\,\mathtt{P}\,| \geq 2$, *there is a non-equivalence (* siR, saT*)* $\xrightarrow{\mathtt{t}}$ *inducing* $\mathtt{P}$*:*

$$(=) \quad |\,\mathtt{P}\,| = 2 \qquad \Rightarrow \qquad \xrightarrow{\mathtt{t}}\colon\ \mathtt{Smm} \qquad \textit{(symmetric);}$$

$$(>) \quad |\,\mathtt{P}\,| > 2 \qquad \Rightarrow \qquad \xrightarrow{\mathtt{t}}\colon\ \mathsf{saS} \qquad \textit{(strong anti-equivalence).}$$

♮

Proof.
Previous results:
Lemma 5.1 (Finite Partitions, p. 44) and
Proposition 5.1 (Permutation for infinite partition, p. 72).

⊣

A.5 Details on Relations Inducing Partitions (Sct. 6)

Theorem 6.1 (Relation inducing partition). *Partition* $\mathtt{P}$ *can be induced by a non-equivalence iff* $|\,\mathtt{P}\,| > 1$.

♮

Proof.

Proposition 4.1 (Relation for one-block partition, p. 65) and

Theorem 5.1 (Relation for non-trivial partition, p. 74).

⊣

Theorem 6.2 (Summary). *Relation inducing* $\mathtt{P} = \{\mathrm{B}_{\mathtt{i}} \subseteq \mathtt{S} \,/\, \mathtt{i} \in \mathtt{I}\}$.

Proof.

Previous results:

Theorem 5.1 (Relation for non-trivial partition, p. 74),

Lemma 5.1 (Finite partitions, p. 44) and

Proposition 4.1 (Relation for one-block partition, p. 65).

⊣

A.6 Details on Relations and Partitions

When does a relation induce a given partition?

Proposition A.1 (Relation and partition). *Given relation* $\longrightarrow$ *and partition* $\mathtt{P}$ *on set* $S \neq \emptyset$, $\longrightarrow\ \lhd\ \mathtt{P}$ *iff* $\longrightarrow$ *and* $\mathtt{P}$ *satisfy the 3 conditions:*

(δ) $S \subseteq \mathrm{Dmn}(\longrightarrow)$ $\qquad \forall a \in S\ \exists b \in S :\ a \longrightarrow b$

(ι) $S \subseteq \mathrm{Img}(\longrightarrow)$ $\qquad \forall b \in S\ \exists a \in S :\ a \longrightarrow b$

(μ) $\forall a \in S\ \forall B \in \mathtt{P} :\ [a] \cap B \neq \emptyset \quad \Rightarrow \quad [a] = B$

♮

Proof. Proposition 2.1: Quotient and partition, p. 20.

($\Rightarrow$) Quotient $S/\!\to$ is a partition $\therefore$ (δ) & (ι)

Given $a \in S$, for some $C \in \mathtt{P}$, $[s] = C$. So:

$[a] \cap B \neq \emptyset \;\Rightarrow\; C \cap B \neq \emptyset \;\Rightarrow\; C = B \;\Rightarrow\; [a] = B$

($\Leftarrow$) We show: (δ) & (ι) & (μ) $\;\Rightarrow\;$ $S/\!\to \; = \mathtt{P}$.

($\subseteq$) Given $a \in S$, by (δ), have some $b \in S$, s. t. $a \to b$; so $b \in [a] \cap \mathtt{P}(b) \neq \emptyset$, whence ($\mu$) yields $[a] = \mathtt{P}(b)$.

($\supseteq$) Given $b \in S$, by (ι), have some $a \in S$, s. t. $a \to b$; so $b \in [a] \cap \mathtt{P}(b) \neq \emptyset$, whence ($\mu$) yields $\mathtt{P}(b) = [a]$.

$\dashv$

Given relation $\longrightarrow$ and partition $\mathtt{P}$ on set $\mathrm{S} \neq \emptyset$:

1. $\longrightarrow$ *smooth on* $\mathtt{P}$ iff

$$\forall a, b \in \mathrm{S} \left(\mathtt{P}(a) = \mathtt{P}(b) \quad \Rightarrow \quad [a] = [b] \right)$$

2. $\longrightarrow$ *nice for* $\mathtt{P}$ iff

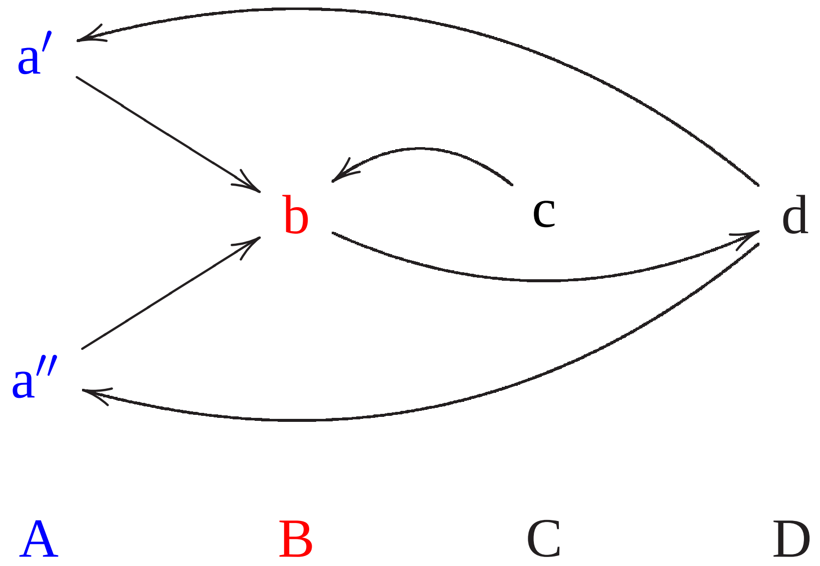
$$\forall a, b \in \mathrm{S} \left([a] = [b] \quad \Rightarrow \quad \mathtt{P}(a) = \mathtt{P}(b) \right)$$

3. $\longrightarrow$ *uniform over* $\mathtt{P}$ iff $\longrightarrow$ smooth on $\mathtt{P}$ & $\longrightarrow$ nice for $\mathtt{P}$.

∂

(See Examples A.6: Smooth relation, p. 80 and A.7: Nice relation, p. 81.)

Example A.6 (Smooth relation)**.**



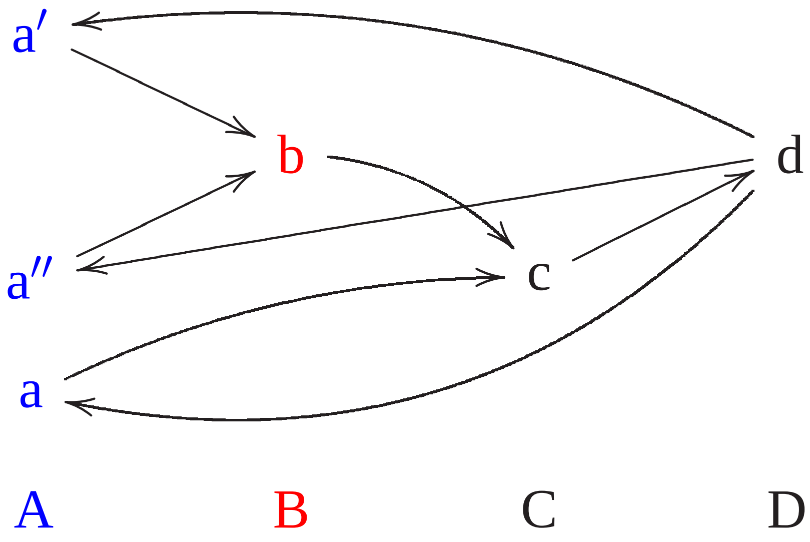
Classes:

$$[a'] = [a''] = B = [c] \qquad [b] = D \qquad [d] = A$$

Relation $\longrightarrow$*:* *not nice* $([a'] = [c]\ \&\ \mathtt{P}(a') \neq \mathtt{P}(c))$.

♭

Example A.7 (Nice relation)**.**



Classes:

$$[a'] = [a''] = B \qquad [b] = C \qquad [c] = D$$

$$[a] = C \qquad\qquad [d] = A$$

Relation $\longrightarrow$*:* *not smooth* *(*$\mathtt{P}(a') = \mathtt{P}(a)$ *&* $[a'] \neq [a]$*).* ♭

Relation on set $\quad\leftrightarrow\quad$ Transformation on partition

Partition $\mathtt{P}$ of set $S \neq \emptyset$.

$(\rightarrow)$ Given relation $\rightarrow$ on S $\quad\mapsto\quad$ *transformation of* $\rightarrow$:

$$A \ \widehat{\rightarrow}\ B \quad\Leftrightarrow\quad \exists a \in A : [a] = B$$

$(\leftarrow)$ Given transformation $\mathtt{t} \subseteq \mathtt{P} \times \mathtt{P}$ $\quad\mapsto\quad$ *relation of* $\mathtt{t}$:

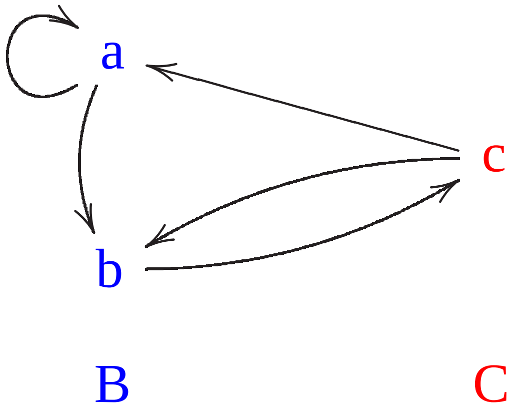
$$a \xrightarrow{\mathtt{t}} b \quad\Leftrightarrow\quad \mathtt{P}(a)^{\mathtt{t}}\ \mathtt{t}\ \mathtt{P}(b)$$

∂

(See Example A.8: Relations and transformations, p. 83.)

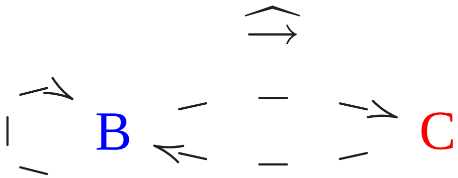
Example A.8 (Relations and transformations). *Set* $\mathrm{S} = \{\,\mathrm{a}, \mathrm{b}, \mathrm{c}\,\}$, *2-block partition* $\mathtt{P} = \{\mathrm{B}, \mathrm{C}\}$, *with* $\mathrm{B} = \{\mathrm{a}, \mathrm{b}\}$ *&* $\mathrm{C} = \{\mathrm{c}\}$ *(cf. Example 3.1: Partition with 2 finite blocks, p. 27).*

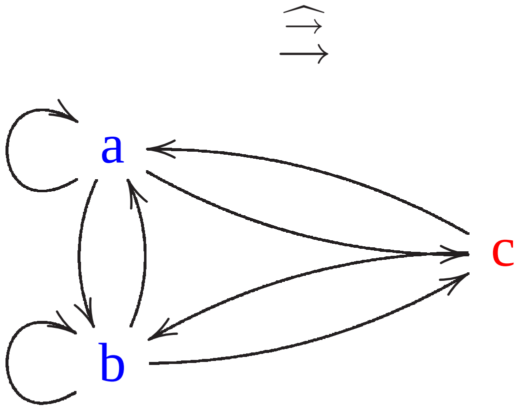
1. *Relation* $\longrightarrow = \{\,(\mathrm{a},\mathrm{a}), (\mathrm{a},\mathrm{b}), (\mathrm{b},\mathrm{c}), (\mathrm{c},\mathrm{a}), (\mathrm{c},\mathrm{b})\,\}$ *(cf. p. 27):*



$[\mathrm{a}] = [\mathrm{c}] = \mathrm{B}$ $\qquad$ $[\mathrm{b}] = \mathrm{C}$

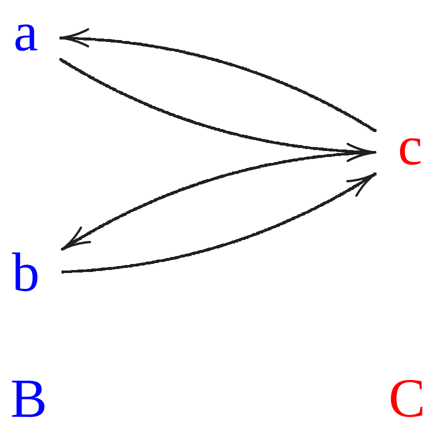
not nice $\qquad$ *not smooth*





$\therefore \quad \widehat{\overrightarrow{\longrightarrow}} \neq \longrightarrow$

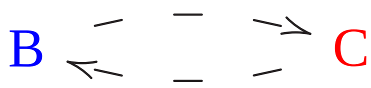
2. Relation $\rightarrow\; = \{(\mathrm{a},\mathrm{c}),(\mathrm{b},\mathrm{c}),(\mathrm{c},\mathrm{a}),(\mathrm{c},\mathrm{b})\}$ *(cf. p. 28):*



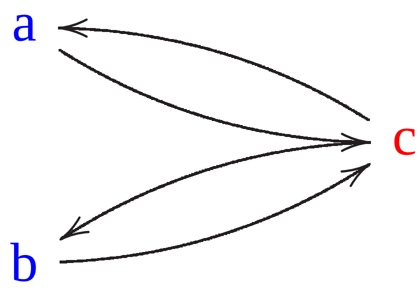
$[\mathrm{a}] = [\mathrm{b}] = \mathrm{C}$ $[\mathrm{c}] = \mathrm{B}$

nice *smooth*

$\widehat{\longrightarrow}$



$\stackrel{\widehat{\rightarrow}}{\longrightarrow}$



$\therefore \quad \stackrel{\widehat{\rightarrow}}{\longrightarrow} = \longrightarrow$

♭

The next result gives some properties of relations for a partition and partition transformations (cf. p. 82).

Proposition A.2 (Relation for partition)**.** *Given partition* $\mathtt{P}$ *and relation* $\rightarrow$ *s. t.* $\rightarrow\ \lhd\ \mathtt{P}$*, consider transformation* $\widehat{\rightarrow} \subseteq \mathtt{P} \times \mathtt{P}$ *and relation* $\overset{\widehat{\rightarrow}}{\rightarrow}$.

1. $\mathrm{Dmn}(\widehat{\rightarrow}) = \mathtt{P} = \mathrm{Img}(\widehat{\rightarrow}) \qquad \rightarrow \subseteq \overset{\widehat{\rightarrow}}{\rightarrow}$.
2. *If* $\rightarrow$ *smooth on* $\mathtt{P}$*, then:* $\overset{\widehat{\rightarrow}}{\rightarrow} \subseteq \rightarrow$ & $\widehat{\rightarrow}$ *is functional.*
3. *If* $\rightarrow$ *nice for* $\mathtt{P}$*, then* $\widehat{\rightarrow}$ *is injective.*

♮

Proof. Proposition A.1: Relation and partition, p. 77.

1. $\mathtt{Dmn}(\widehat{\rightarrow}) = \mathtt{P} = \mathtt{Img}(\widehat{\rightarrow}) \qquad \rightarrow \subseteq \overset{\widehat{\rightarrow}}{\rightarrow}$

   ($\mathtt{Dmn}$) For $A \in \mathtt{P}$, have $a \in A$, so have $B \in \mathtt{P}$ s. t. $[a] = B$, thus $A \widehat{\rightarrow} B$.

   ($\mathtt{Img}$) For $B \in \mathtt{P}$, have $a \in S$ s. t. $[a] = B$, thus $\mathtt{P}(a) \widehat{\rightarrow} B$.

   ($\subseteq$) $a \rightarrow b \;\Rightarrow\; b \in [a] \cap \mathtt{P}(b) \neq \emptyset \;\Rightarrow\; [a] = \mathtt{P}(b) \;\Rightarrow$
   $\exists a \in \mathtt{P}(a) : [a] = \mathtt{P}(b) \;\Rightarrow\; \mathtt{P}(a) \widehat{\rightarrow} \mathtt{P}(b) \;\Rightarrow\; a \overset{\widehat{\rightarrow}}{\rightarrow} b$.

2. $\rightarrow$ smooth on $\mathtt{P} \quad \Rightarrow \quad \overset{\widehat{\rightarrow}}{\rightarrow} \subseteq \rightarrow \quad \& \quad \widehat{\rightarrow}$: functional

   (a) $a \overset{\widehat{\rightarrow}}{\rightarrow} b \;\Rightarrow\; \mathtt{P}(a) \widehat{\rightarrow} \mathtt{P}(b) \;\Rightarrow\; \exists a' \in \mathtt{P}(a) : [a'] = \mathtt{P}(b) \;\Rightarrow$
   $[a] = [a'] \;\Rightarrow\; [a] = [a'] = \mathtt{P}(b) \;\Rightarrow\; a \rightarrow b$.

   (b) $A \widehat{\rightarrow} B' \;\&\; A \widehat{\rightarrow} B'' \;\Rightarrow\; \exists a' \in A : [a'] = B' \;\&\; \exists a'' \in A : [a''] = B'' \;\Rightarrow$
   $[a'] = [a''] \;\Rightarrow\; B' = [a'] = [a''] = B''$.

3. $\rightarrow$ nice for $\mathtt{P} \quad \Rightarrow \quad \widehat{\rightarrow}$: injective
   $A' \widehat{\rightarrow} B \;\&\; A'' \widehat{\rightarrow} B \; \exists a' \in A' : [a'] = B \;\&\; \exists a'' \in A'' : [a''] = B \;\Rightarrow$
   $[a'] = [a''] \;\Rightarrow\; \mathtt{P}(a') = \mathtt{P}(a'') \;\Rightarrow\; A' = \mathtt{P}(a') = \mathtt{P}(a'') = A''$.

$\dashv$

A uniform relation for a partition is obtained from a permutation.

Theorem A.1 (Uniform relation for partition)**.** *Consider partition* $\mathtt{P}$ *and relation* $\rightarrow$ *s. t.* $\rightarrow$ *induces* $\mathtt{P}$ *and* $\rightarrow$ *is uniform over* $\mathtt{P}$.

1. *Transformation* $\widehat{\rightarrow}$ *is a permutation on* $\mathtt{P}$.

2. *Relations coincide:* $\rightarrow \; = \; \overset{\widehat{\rightarrow}}{\rightarrow}$.

3. *Hence: relation* $\rightarrow$ *is the relation* $\overset{\widehat{\rightarrow}}{\rightarrow}$ *of permutation* $\widehat{\rightarrow}$ *on* $\mathtt{P}$.

$\natural$

Proof. Proposition A.2: Relation for partition, p. 85.

$\dashv$