A Tableau Calculus for Hybrid XPath with Data

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  - then, introduce XPath,
  - and XPath with Data,
  - and Hybrid XPath with Data,
  - and then, introduce the promised Tableau Calculus

Tableaux for Modal Logics

Start with a tableaux for propositional logic.

$$\frac{(\varphi \wedge \psi)}{\begin{array}{c}\varphi \\ \psi\end{array}} \ (\wedge)$$

$$\frac{\neg(\varphi \wedge \psi)}{\neg\varphi \qquad \neg\psi} \ (\neg\wedge)$$

$$\frac{\neg\neg\varphi}{\varphi} \ (\neg\neg)$$

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- Neat!: 3 rules for an NP-complete problem!
- But now we have to perform boolean reasoning in each point of the model.
- The solution is: labels

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Remember the semantics for $\langle R \rangle$

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  If there is an R-successor t, then φ should not hold at t.

$$\frac{\begin{array}{l} s{:}\neg\langle R \rangle \varphi \\ sRt \end{array}}{t{:}\neg\varphi} \ (\neg\langle R \rangle)$$

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- To check satisfiability of φ, start the tableaux with the labelled formula $s{:}\varphi$.
- The previous 5 rules provide a sound and complete calculus for the basic modal logic
- It is actually terminating, and by imposing some restrictions on applications it can run in PSPACE, so it is optimal.
- But what is the status of labeled formulas $s{:}\varphi$ and accessibility statements sRt?
- Enter Hybrid Logics ...

Hybrid Logics

- The first ideas behind hybrid logics go back to Arthur Prior, but its modern development flourish with the work of Blackburn and Seligman.
- For this talk we can think of hybrid logics as a way to add constants (and equality) to modal languages, so that we can internalize labelled deduction.

Blackburn, P.
Internalizing Labelled Deduction
Journal of Logic and Computation, 10, 137-168, 2000.

Nominals and Satisfaction Operators

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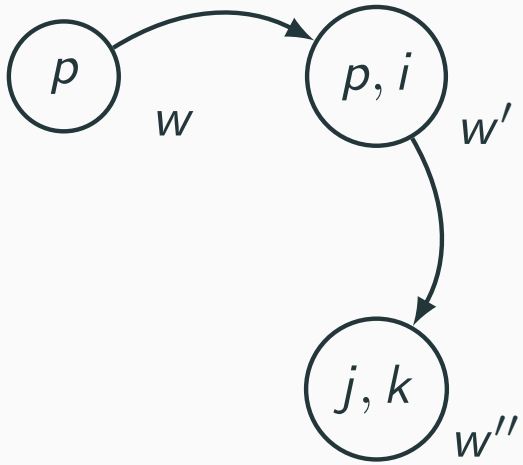

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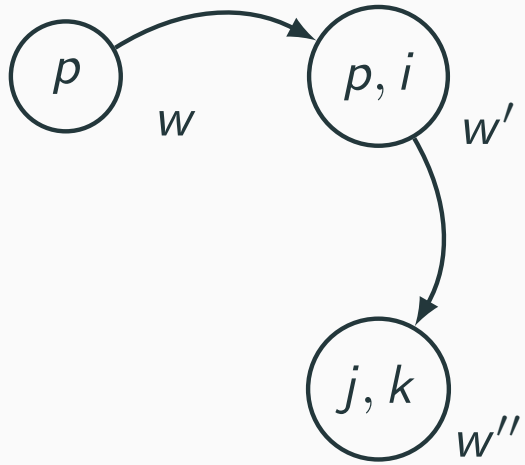

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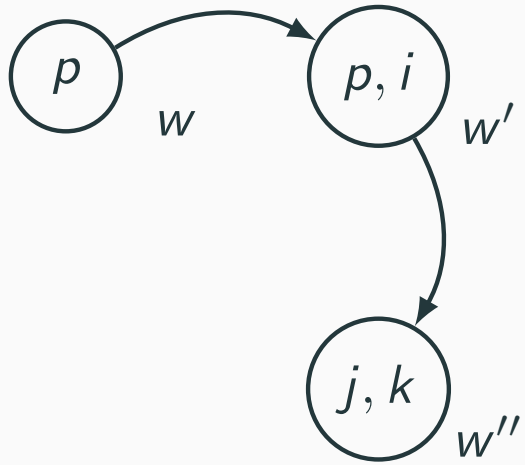

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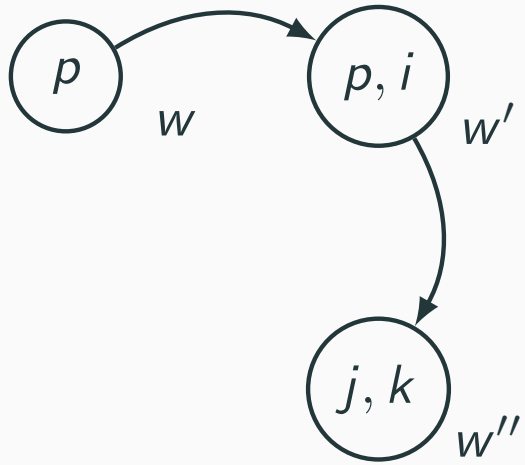

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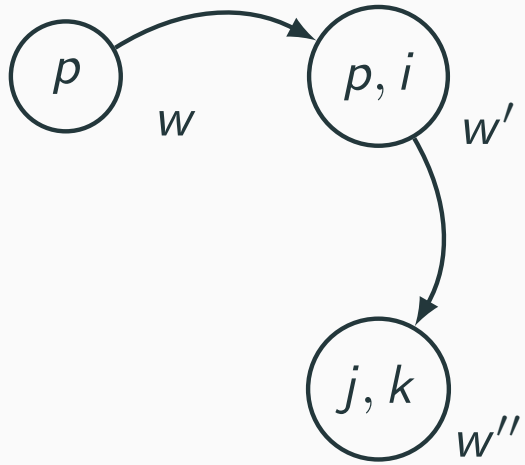

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- We can internalize labelled deduction.

$$\frac{i{:}\langle R \rangle \varphi}{\begin{array}{c} i{:}\langle R \rangle j \\ j{:}\varphi \end{array}} (\langle R \rangle) \qquad (j) \text{ a new nominal} \qquad \frac{\begin{array}{c} i{:}\neg\langle R \rangle \varphi \\ i{:}\langle R \rangle j \end{array}}{j{:}\neg\varphi} (\neg\langle R \rangle)$$

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- As a plus, the language has (limited) equality

$$\frac{}{i{:}i} (Ref) \qquad \frac{i{:}j}{j{:}i} (Sym) \qquad \frac{i{:}\varphi \quad i{:}k \quad j{:}k}{j{:}\varphi} (Cong)$$

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- (Start the tableau with $i{:}\varphi$ for i not in φ).

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$$\frac{i{:}\varphi \;\; i{:}k \;\; j{:}k}{j{:}\varphi}\,(wCong^{1}) \qquad \frac{i{:}n \;\; i{:}k \;\; j{:}k}{j{:}n}\,(Nom)$$

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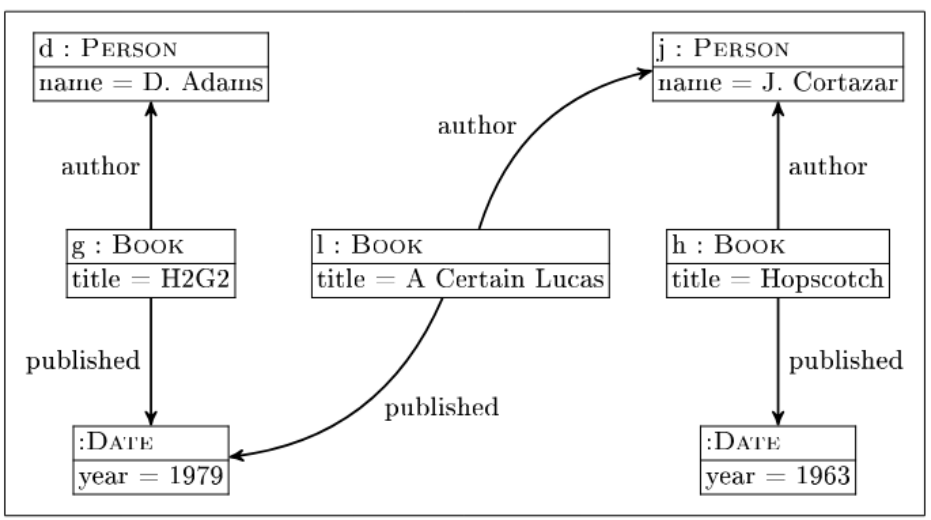
Bolander, T. and Blackburn, P..
Termination for Hybrid Tableaus
Journal of Logic and Computation, 17, 517-554, 2007.

Semi-Structured Databases

- Many data-intensive applications use complex data that is not naturally encoded in a relational database.
  - Web data or biological data are better described using semi-structured data models, that organize information as labeled trees or graphs.
  - They contain labels from a finite alphabet (the structural information), and from an infinite alphabet (the actual data).
- A well known example is XML (eXtensible Markup Language), the most successful data model of this kind.
- XPath is the most widely used XML query language.

Consider the following picture that shows the information on a part of a semi-structured database containing details about books.



From the information in the picture we can infer that D. Addams and J. Cortazar published a book the same year (on 1979).

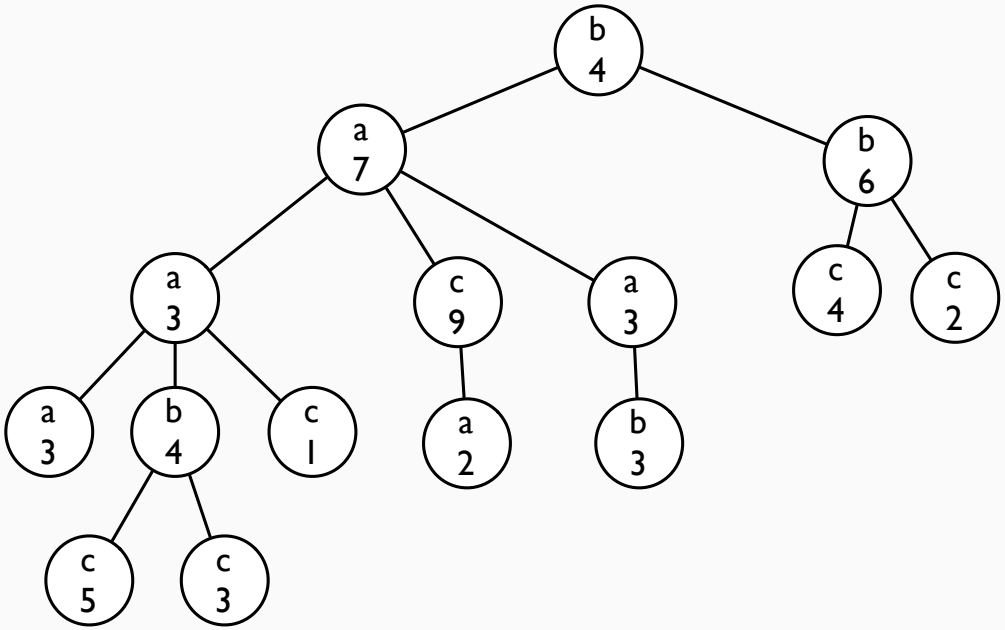
XPath with Data Tests

Data tree

- ordered tree (finite or infinite)
- nodes have
  - a label (finite alphabet)
  - a data (infinite domain)

Path expressions

- go to ancestor, check *a*, go to child, check *c*, go right sibling, go to descendant, check *b*



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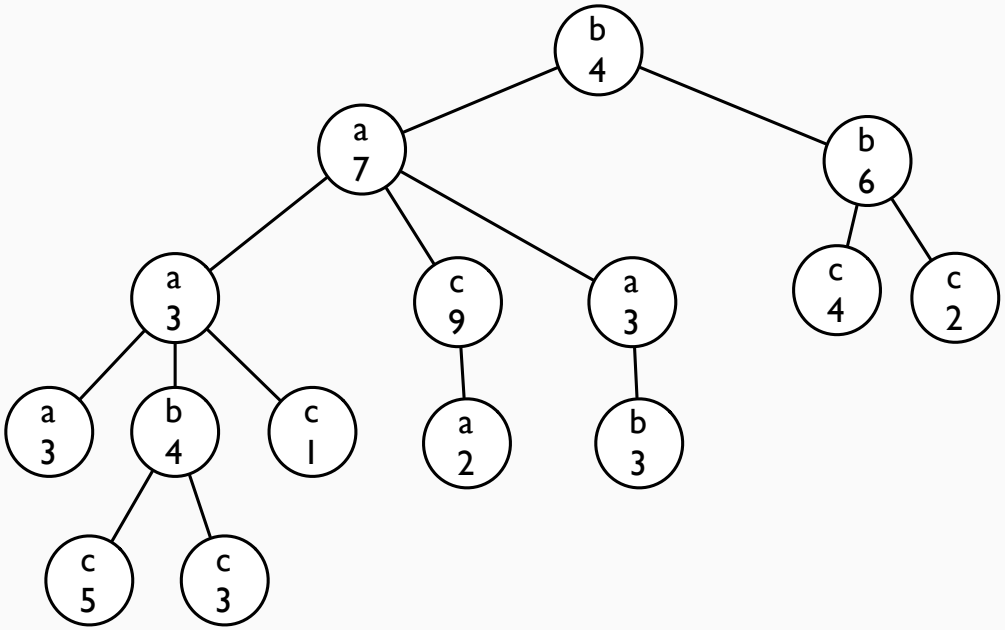

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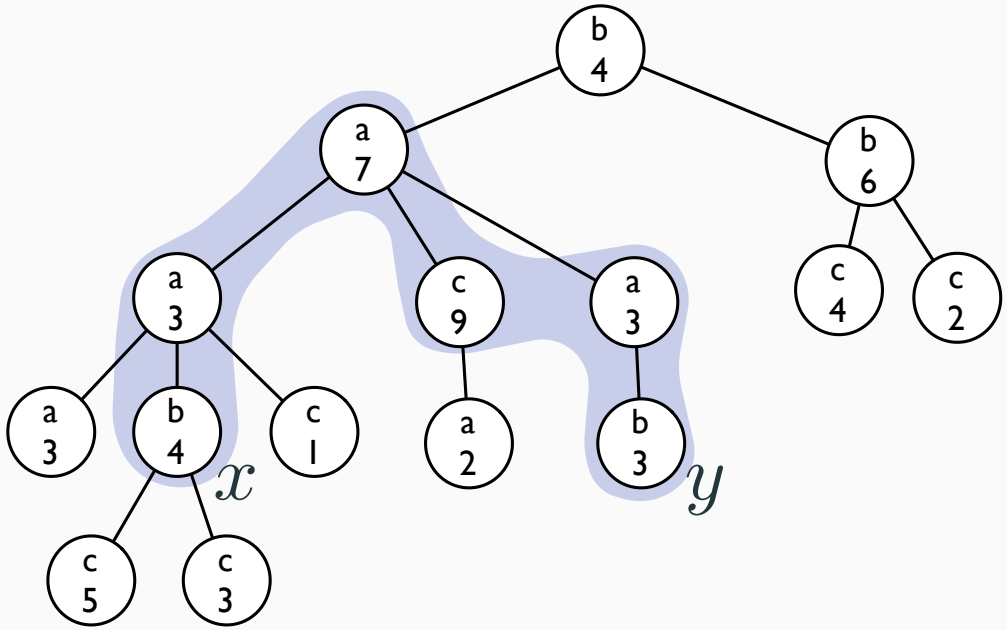

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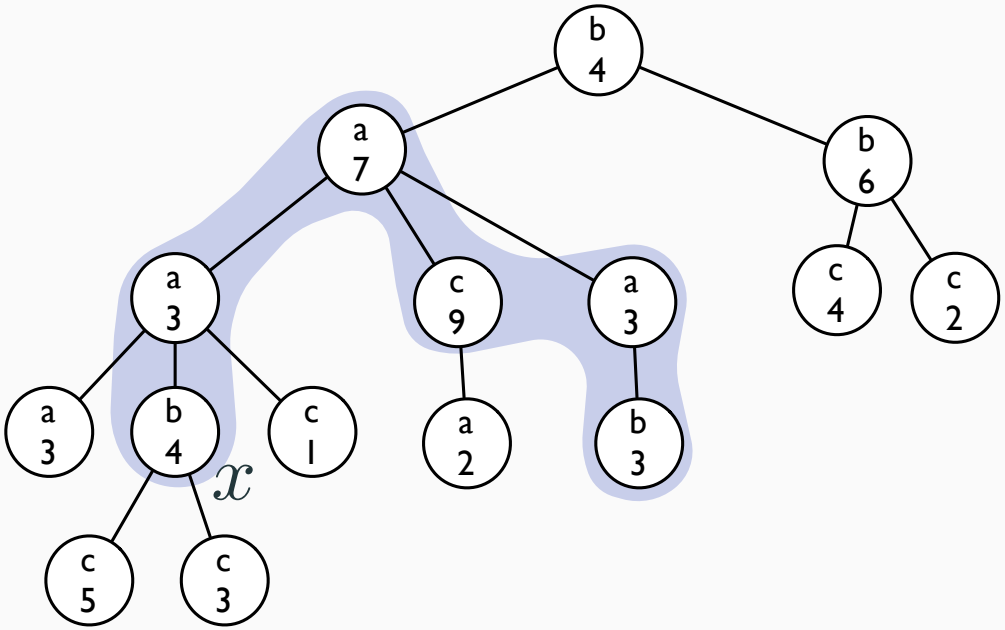

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XPath with Data Tests

Data tree

- ordered tree (finite or infinite)
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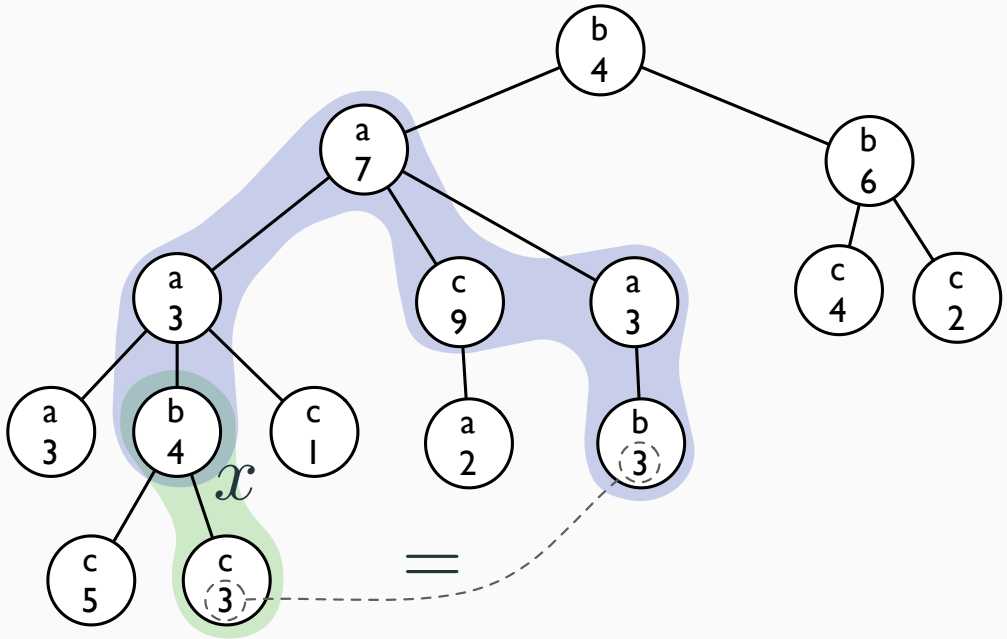

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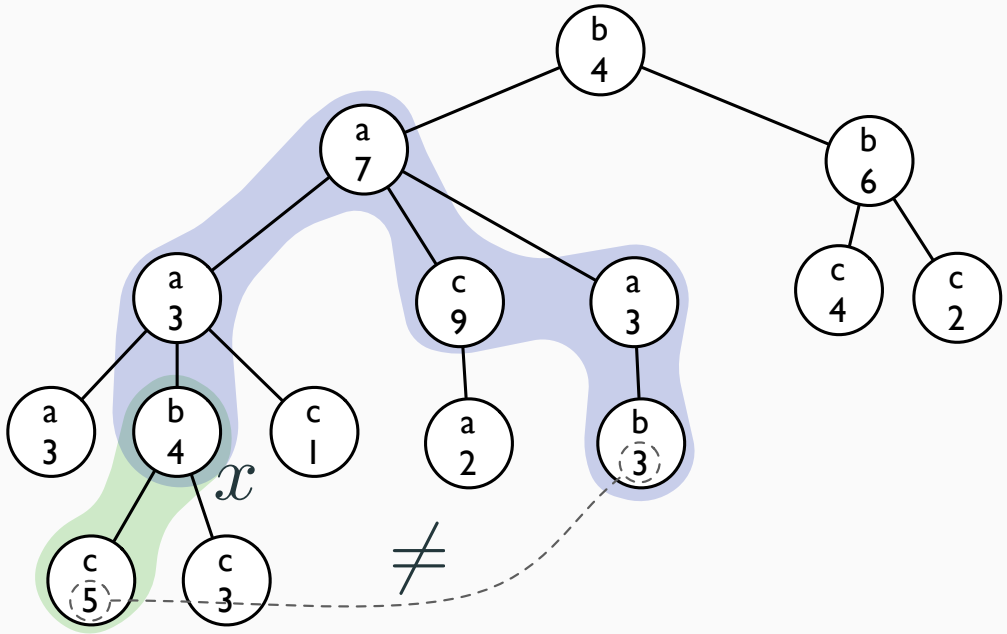

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Formally. . .

- Node and Path Expressions

$$\alpha, \beta \;:=\; \delta \mid [\varphi] \mid \alpha\beta \mid \alpha \cup \beta$$
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where $p \in \mathsf{Prop}$, $\delta \in \{\downarrow, \uparrow, \rightarrow, \downarrow^*, \uparrow^*\}$, $* \in \{=, \neq\}$.

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\mathcal{M}, x, y \models [\varphi] &\iff x = y \text{ and } \mathcal{M}, x \models \varphi \\
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\mathcal{M}, x, y \models \alpha \cup \beta &\iff \mathcal{M}, x, y \models \alpha \text{ or } \mathcal{M}, x, y \models \beta
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(But, of course, the language is more expressive).

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(notice that the second i: is different from the first one).

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- And actually

$$i{:}\varphi \leftrightarrow \langle @_i[\varphi] = @_i[\varphi] \rangle$$

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Some HXPath$_=$ expressions together with their intuitive meaning:

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$\langle @_i = @_j \rangle$	The node named i has the same data than the node named j.
$\langle \alpha = @_i\beta \rangle$	A node accessible from the current node by an α path has the same data than a node accessible from the point named i by a β path.

Tableau Rules: Boolean, @ and Nominals

$$\frac{n{:}\neg\neg\varphi}{n{:}\varphi}\ (\neg\neg) \qquad \frac{n{:}(\varphi \wedge \psi)}{\begin{array}{l} n{:}\varphi \\ n{:}\psi \end{array}}\ (\wedge) \qquad \frac{n{:}\neg(\varphi \wedge \psi)}{n{:}\neg\varphi \quad \mid \quad n{:}\neg\psi}\ (\neg\wedge)$$

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$$\frac{i{:}@_j\varphi}{j{:}\varphi}\ (@) \qquad \frac{i{:}\neg@_j\varphi}{j{:}\neg\varphi}\ (\neg@)$$

$$\frac{}{n{:}n}\ (Ref^2) \qquad \frac{n{:}i}{i{:}n}\ (Sym)$$

$$\frac{n{:}\varphi \quad n{:}i \quad m{:}i}{m{:}\varphi}\ (wCong^3) \qquad \frac{n{:}j \quad n{:}i \quad m{:}i}{m{:}j}\ (Nom)$$

[2] n appears in the tableau.

[3] m is the smallest nominal making i true in the tableau.

Tableau Rules: Prefix

$$\frac{n{:}\langle \alpha * \beta \rangle}{\langle @_n \alpha * @_n \beta \rangle}\ (\mathit{Int}) \qquad \frac{n{:}\neg\langle \alpha * \beta \rangle}{\neg\langle @_n \alpha * @_n \beta \rangle}\ (\neg\mathit{Int}) \qquad \frac{n{:}m}{\langle @_n = @_m \rangle}\ (\mathit{IData})$$

- The satisfaction operator is moved from the node formula to the path formula to anchor each path.
- Notice that the result is an (unlabelled) XPath diamond.
- It only remains to give rules for the different path operators that can occur inside a diamond.

Tableau Rules: Down

$$\frac{\langle @_n{\downarrow}\alpha * \beta\rangle}{\begin{array}{c} n \to m \\ \langle @_m\alpha * \beta\rangle \end{array}} (\downarrow^1) \qquad \frac{\neg\langle @_n{\downarrow}\alpha * \beta\rangle \quad n \to m}{\neg\langle @_m\alpha * \beta\rangle} (\neg\downarrow)$$

[1] m is the smallest nominal that has not appeared in the tableau.

Tableau Rules: Tests

$$\frac{\langle @_n[\varphi]\alpha * \beta\rangle}{\begin{array}{c} n{:}\varphi \\ \langle @_n\alpha * \beta\rangle \end{array}}\,([\,]) \qquad \frac{\neg\langle @_n[\varphi]\alpha * \beta\rangle}{n{:}\neg\varphi \;\mid\; \neg\langle @_n\alpha * \beta\rangle}\,(\neg[\,])$$

Tableau Rules: Union

$$\frac{\langle @_n(\alpha \cup \beta)\gamma * \delta\rangle}{\langle @_n\alpha\gamma * \delta\rangle \mid \langle @_n\beta\gamma * \delta\rangle}\ (\cup) \qquad \frac{\neg\langle @_n(\alpha \cup \beta)\gamma * \delta\rangle}{\begin{array}{c}\neg\langle @_n\alpha\gamma * \delta\rangle \\ \neg\langle @_n\beta\gamma * \delta\rangle\end{array}}\ (\neg\cup)$$

Tableau Rules: At

$$\frac{\langle @_n @_i \alpha * \beta \rangle}{\langle @_i \alpha * \beta \rangle}\ (@) \qquad \frac{\neg\langle @_n @_i \alpha * \beta \rangle}{\neg\langle @_i \alpha * \beta \rangle}\ (\neg @)$$

Tableau Rules: Data

$$\frac{\neg\langle @_n \neq @_m \rangle}{\langle @_n = @_m \rangle}\,(DEq)$$

$$\frac{\langle @_n = @_m \rangle}{\langle @_m = @_n \rangle}\,(DSym)$$

$$\frac{\langle @_n = @_m \rangle \;\; \langle @_m = @_k \rangle}{\langle @_n = @_k \rangle}\,(DTrans)$$

Tableau Rules: Commutativity

$$\frac{\langle @_n * \varphi \rangle}{\langle \varphi * @_n \rangle}\ (\text{COM})$$

$$\frac{\neg\langle @_n * \varphi \rangle}{\neg\langle \varphi * @_n \rangle}\ (\neg\text{-COM})$$

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  To check the satisfiability of α start the tableaux with $i{:}\langle\alpha\rangle\top$
- A branch Θ of a tableau contains a clash if one of the following conditions holds:
  1. $\{n{:}a, n{:}\neg a\} \subseteq \Theta$
  2. $\langle @_n \neq @_n\rangle \in \Theta$,
  3. $\neg\langle @_n = @_n\rangle \in \Theta$,
  4. $\{\langle @_n = @_m\rangle, \langle @_n \neq @_m\rangle\} \subseteq \Theta$,
  5. $\{\langle @_n = @_m\rangle, \neg\langle @_n = @_m\rangle\} \subseteq \Theta$,

  for some $n,\ m \in \mathrm{NOM}$, $a \in \mathrm{NOM} \cup \mathrm{PROP}$.

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  - Now looking at transitive closure, siblings, and other data comparison (e.g., $\leq$).
- Implementations.

Future Work

- Interpolation.
- Probabilistic models.
- Data aggregation (max, avg, sum)
- We have a nice sequent calculus, what about natural deduction?
- Transitive closure over trees.