Truth-tables for Intuitionistic Propositional Logic

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Outline

1. Why?

2. Gödel Limiting Result

3. A new approach

Why Intuitionistic Logic?

- Classical logic assumes:
    - Law of Excluded Middle: $P \vee \neg P$
    - Proof by contradiction
- Intuitionism (Brouwer):
    - Mathematics as mental construction
    - Truth requires *explicit evidence*
- Logical meaning is tied to *proofs*, not truth values

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Constructive View of Truth

Key Principle

A proposition is true *iff* we have a construction (proof) of it.

- $\exists x. P(x)$ means we can compute such an x
- $P \vee Q$ means we know which one holds
- $\neg P$ means $P \to \bot$

BHK Semantics

- $P \wedge Q$: a proof of P *and* a proof of Q
- $P \vee Q$: either a proof of P or a proof of Q
- $P \rightarrow Q$: a construction mapping proofs of P to proofs of Q
- $\perp$: no proof
- $\neg P$: $P \rightarrow \perp$

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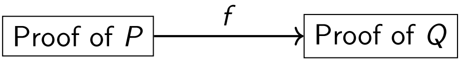
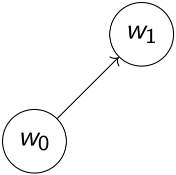
Implication as a Function



Interpretation

A proof of $P \to Q$ is a *function* mapping proofs of P to proofs of Q.

Why $P \vee \neg P$ May Fail



- At w_0, neither P nor $\neg P$ is known
- P may become true later
- So $w_0 \not\Vdash P \vee \neg P$

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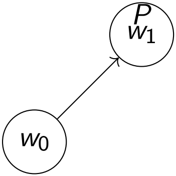

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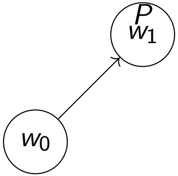

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Logic = Types, Proofs = Programs

Curry–Howard Correspondence

$$\text{Propositions} \leftrightarrow \text{Types}, \quad \text{Proofs} \leftrightarrow \text{Programs}$$

- $P \wedge Q \leftrightarrow$ product types
- $P \vee Q \leftrightarrow$ sum types
- $P \to Q \leftrightarrow$ function types

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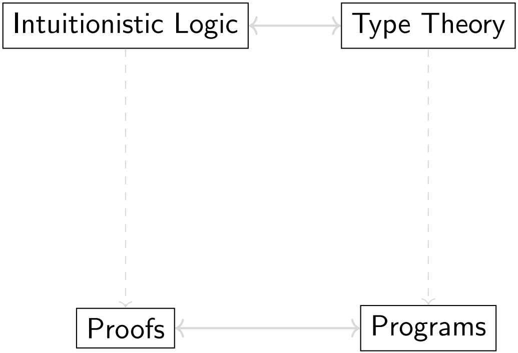

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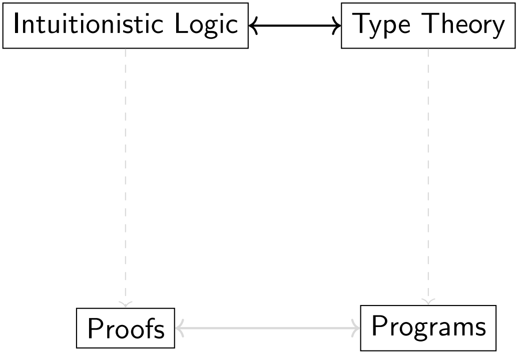

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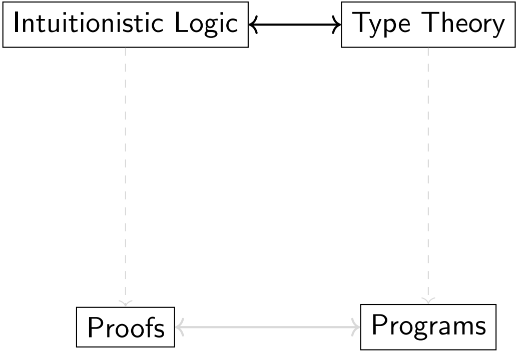

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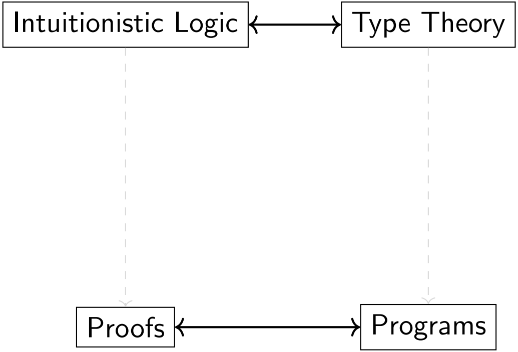

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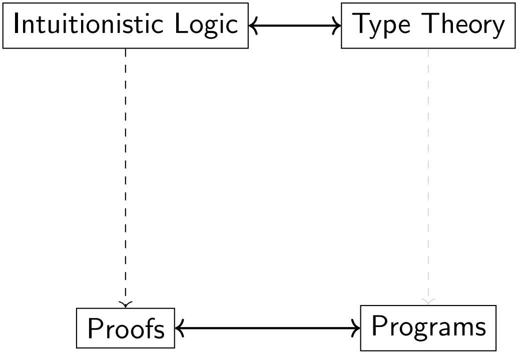

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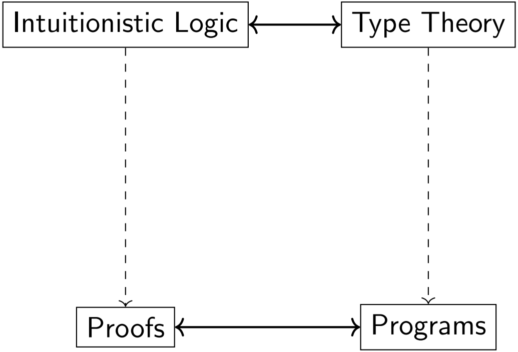


Gödel-Gentzen Double Negation Translation

Every intuitionistic formula A can be translated into a classical formula A^N:

$$P^N = \neg\neg P$$
$$(A \wedge B)^N = A^N \wedge B^N$$
$$(A \vee B)^N = \neg(\neg A^N \wedge \neg B^N)$$
$$(A \rightarrow B)^N = A^N \rightarrow B^N$$

Result
A is intuitionistically provable iff A^N is classically provable.

Intuitionistic Logic as Modal Logic S4

Gödel showed:

$$\text{Intuitionistic Logic} \subseteq \text{Modal Logic S4}$$

Translation:

- Atomic P becomes $\Box P$
- $A \to B$ becomes $\Box(A \to B)$

Intuition

Truth is stable under future information.

Kripke Models for Intuitionistic Logic

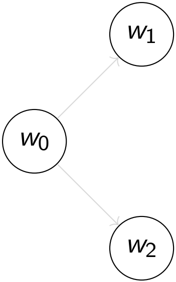
A Kripke model consists of:

- A partially ordered set $(W, \leq)$
- Monotone valuation:

$$w \Vdash P \Rightarrow v \Vdash P \quad \text{for } w \leq v$$

Idea
Knowledge grows; truths persist.

Kripke Model: Worlds and Accessibility



Accessibility

$w_0 \leq w_1$ and $w_0 \leq w_2$ represent growth of information.

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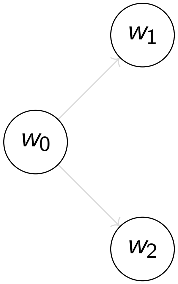

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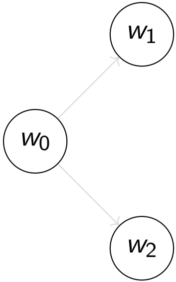

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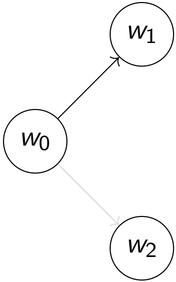

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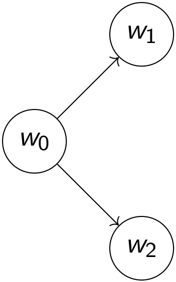

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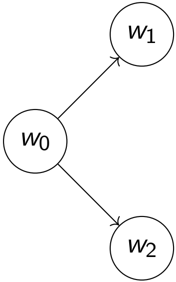

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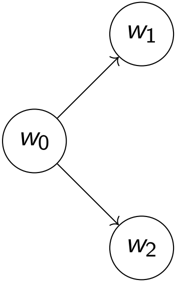
Monotonicity of Truth



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- $w_1 \Vdash P$
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Persistence
Truth propagates forward in the order.

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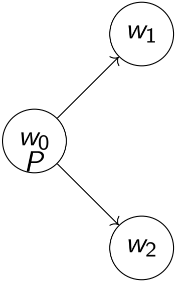

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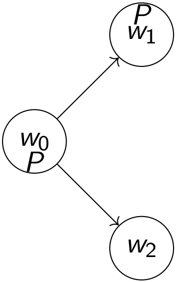

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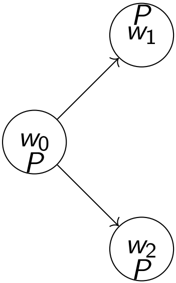

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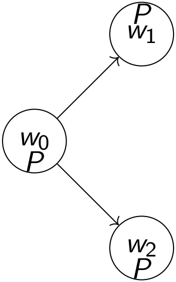

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No Finite Truth Tables

Gödel's Result

Intuitionistic logic cannot be characterized by any finite truth table semantics.

- Classical logic: 2 truth values
- Many-valued logics: finitely many values
- Intuitionistic logic requires *infinite semantic structure*

Consequence

Truth is not static — it depends on informational growth.

Summary

- Intuitionistic logic emphasizes constructions
- BHK semantics explains meaning via proofs
- Curry–Howard connects logic to computation
- Gödel translations embed intuitionism into classical and modal logics
- Kripke models capture evolving knowledge

From Truth Tables to NMatrices

- In classical logic:
  - Each connective has a *truth table*
  - Each input has exactly *one* output
- Many logics cannot be captured this way
- Idea: allow *multiple possible outputs*

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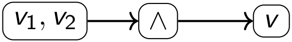

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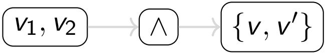

NMatrix

Key Difference

Truth tables return *one value*; NMatrices return a *set of admissible values*.

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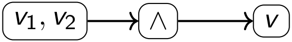

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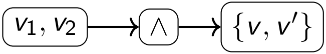

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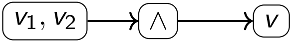

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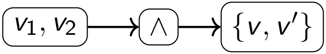

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What NMatrices Capture

- NMatrices generalize truth tables
- They allow:
  - Finite sets of truth values
  - Non-deterministic connectives
- They work well for:
  - Paraconsistent logics
  - Paracomplete logics
  - Many substructural logics

Crucial Assumption

Semantic evaluation is still *local*:

value of A depends only on values of subformulas

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Why Intuitionistic Logic Is Different

- Intuitionistic truth is not static
- Truth depends on:
    - Possible future information
    - Extensions of the current state
- In Kripke semantics:

$$w \Vdash A \to B \text{ iff } \forall v \geq w,\ v \Vdash A \Rightarrow v \Vdash B$$

Key Observation

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Why NMatrices Fail for Intuitionistic Logic

- NMatrices assign values compositionally
- Intuitionistic implication depends on:
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  - Persistence of truth
- No finite set of values can encode:
  - All possible future behaviors
  - All Kripke countermodels

Theorem

There is no finite NMatrix that is sound and complete for full intuitionistic logic.

Intuition

Non-determinism is not enough — intuitionistic logic needs *semantic growth*.

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From NMatrices to RNMatrices

- NMatrices evaluate formulas locally
- Intuitionistic implication is global
- Idea: keep non-determinism *and* add constraints

RNMatrix (Informal Idea)

An RNMatrix is an NMatrix together with *restrictions on which non-deterministic choices are allowed*.

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Not every valuation is admissible — only those respecting logical structure.

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RNMatrices: Semantics with Restrictions

- Start with an NMatrix:

$$\mathcal{M} = \langle V, D, \mathcal{O} \rangle$$

- Add a set of *semantic restrictions* $\mathcal{R}$

RNMatrix

$$\mathcal{M}_R = \langle V, D, \mathcal{O}, \mathcal{R} \rangle$$

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Role of the Restrictions

- Restrictions relate values across formulas
- They enforce properties like:
  - Persistence of truth
  - Behavior of implication
  - Interaction between connectives
- Example (Informal):
  "If $A \to B$ is designated and A is designated, then B must be designated."

Key Difference

Truth is no longer purely truth-functional.

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- Restrictions relate values across formulas
- They enforce properties like:
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  - Behavior of implication
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NMatrices vs RNMatrices

	NMatrices	**RNMatrices**
Evaluation	Local	Constrained
Non-determinism	Free	Restricted
Truth-functional	Yes	No
Semantic Power	Limited	Higher

Why Level Valuations?

- RNMatrices restrict admissible valuations
- Restrictions must talk about:
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Syntactic Level

The *level* of a formula is its complexity:

- Atomic formulas: level 0
- $A \circ B$: level $1 + \max(\ell(A), \ell(B))$

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A *level valuation* assigns truth values *level by level*, respecting RNMatrix restrictions.

- First assign values to atoms
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- Sometimes not all formulas are assigned yet
- A *partial level valuation:*
  - Assigns values up to level n
  - Leaves higher levels undefined

Purpose

Allows checking restrictions locally before extending the valuation.

Intuition

Like building a Kripke model one layer at a time.

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- When assigning a value at level $n+1$
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Every extension of a partial level valuation preserves admissibility.

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Future choices are constrained by past truths.

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- Level valuations respect syntactic structure
- Partial valuations control semantic growth
- Restrictions encode intuitionistic persistence

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RNMatrices + level valuations simulate Kripke semantics without explicit worlds.

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What Must Be Captured: S4

- Modal logic S4 is characterized by:
  - Reflexivity: $\Box A \to A$
  - Transitivity: $\Box A \to \Box\Box A$
- Kripke semantics:
  - Worlds partially ordered by accessibility
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The modality $\Box$ is *global*: it quantifies over all accessible worlds.

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Encoding Worlds via Levels

- RNMatrices do not use explicit worlds
- Instead, they use:
  - Levels of valuation
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- Interpretation:
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Analogy

Worlds $\longleftrightarrow$ Levels of valuation

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RNMatrix Encoding of $\Box$

- In Kripke semantics:

$$w \Vdash \Box A \iff \forall v \geq w,\ v \Vdash A$$

- In RNMatrices:
  - $\Box A$ is evaluated at a given level n
  - Restrictions force A to hold at all higher levels

Restriction (Informal)

If $\Box A$ is designated at level n, then A must be designated at every level $m \geq n$.

Effect

The restriction replaces universal quantification over worlds.

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- Reflexivity:
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  - So $\Box A \rightarrow A$ holds
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RNMatrix restrictions encode exactly the preorder structure of S4: truth-tables for S4!

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A Canonical Test Formula

Formula

$$P \vee \neg P$$

- Law of Excluded Middle
- Valid classically
- Not valid intuitionistically

Goal

Show how an RNMatrix admits a valuation where $P \vee \neg P$ is *not designated.*

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Level Valuation for P

- Consider a partial level valuation
- Assign values to P at increasing levels

Level	P
0	0
1	1
2	1

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- Evaluate at level 0

Value of P

$$P = 0$$

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- $\neg P$ means $P \to \bot$
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- At level 1: $P = 1$

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$\neg P$ is **not designated** at level 0.

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$A \vee B$ is designated only if at least one disjunct is designated.

Result

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- RNMatrices admit future truth growth
- Negation is globally constrained
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Let's use S4 translation!!!

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We Can Do Better!

A proper level valuation function for IPL

As P is translated into **S4** as $\Box P$, we may try using a pair of truth values $\langle n, m \rangle$ such that $0 \leq n, m \leq 2$.

Example: testing $P \vee \neg P$:

$\langle p \; \Box p \rangle$	$\langle \neg(\Box p)$	$\Box\neg(\Box p)\rangle$	$\langle \Box p \vee \Box\neg(\Box p)$	$\Box(\Box p \vee \Box\neg(\Box p))\rangle$
$\langle 0 \; 0 \rangle$	$\langle 1$	$0 \rangle$	$\langle 0$	$0 \rangle$
$\langle 0 \; 0 \rangle$	$\langle 2$	$2 \rangle$	$\langle 2$	$2 \rangle$
$\langle 2 \; 2 \rangle$	$\langle 0$	$0 \rangle$	$\langle 2$	$2 \rangle$

Observe that the first value of the second column of first row (1) must be validated by another row having 0; this implies, in IPL, that every 1 must be validated by a 0. This lead us to the following:

Level valuations for IPL

Definition

Let $Val(\mathcal{M}_{IPL})$ be the set of valuations over $\mathcal{M}_{IPL}$. We define the level $\mathcal{L}_k^{IPL}$, where $k \in \mathbb{N}$, as follows:

1. $\mathcal{L}_0^{IPL} = Val(\mathcal{M}_{IPL}) \setminus \{1\}$;
2. $\mathcal{L}_{k+1}^{IPL} = \{v \in \mathcal{L}_k^{IPL} \mid \forall\alpha \in \Theta_{k+1}$, if $v(\alpha) = \mathbf{1}$ then there exists $w \in \mathcal{L}_k^{IPL}$ such that $w(\alpha) = \mathbf{0}$ and: $v(\beta) = \mathbf{2}$ implies $w(\beta) = \mathbf{2}$ for every $\beta \in \Theta\}$.

Definition

The set of level valuations in $\mathcal{M}_{IPL}$ is defined as the intersection of every level over $\mathcal{M}_{IPL}$:

$$\mathcal{L}_{IPL} = \bigcap_{k \geq 0}^{\infty} \mathcal{L}_k^{IPL}$$

Intuitionistic Partial (Level) Valuations

Let Λ a set closed under subformulae. A *partial valuation in* $\mathcal{M}_{IPL}$ is a function $\tilde{v}_p : \Lambda \to V_{IPL}$ such that, for every $\alpha, \beta \in \Lambda$:

- if $\alpha \in \mathcal{P} \cap \Lambda$ then $\tilde{v}_p(\alpha) \in Val(\mathcal{M}_{IPL}) \setminus \{1\}$
- if $\neg\alpha \in \Lambda$ then $\tilde{v}_p(\neg\alpha) \in \neg^{IPL}(\tilde{v}_p(\alpha))$
- if $\# \in \{\to, \vee, \wedge\}$ and $\alpha\#\beta \in \Lambda$ then $\tilde{v}_p(\alpha\#\beta) \in \#^{IPL}(\tilde{v}_p(\alpha), \tilde{v}_p(\beta))$

The set of partial valuations in $\mathcal{M}_{IPL}$ with domain Λ is $iPV(\Lambda)$.

Intuitionistic Partial (Level) Valuations

$$iPLV(\Lambda) = \{\tilde{v}_p \in iPV(\Lambda) \mid \forall \alpha \in \Lambda\big(\tilde{v}_p(\alpha) = 1 \text{ implies that } \tilde{w}_p(\alpha) = 0 \text{ and, } \forall \beta \in \Lambda,\ \tilde{v}_p(\beta) = 2 \Rightarrow \tilde{w}_p(\beta) = 2, \text{ for some } \tilde{w}_p \in iPLV(\Lambda)\big)\}$$

Since $iPV(\Lambda)$ is finite, the definition of $iPLV(\Lambda)$ is not cyclic.
An *intuitionistic truth table in* $\mathcal{M}_{IPL}$ is an exhaustive list of all intuitionistic partial level valuations for a given finite domain closed under subformulas.
Each of these partial valuations corresponds to a row of the table.

Example in IPL

Figure: Initial table

Row (ID)	p	$\neg p$	$p \vee \neg p$	$\neg(p \vee \neg p)$	$\neg\neg(p \vee \neg p)$
(1)	**F**	**U**	**F**	**U**	**U**
(2)	**F**	**U**	**F**	**U**	**T**
(3)	**F**	**U**	**F**	**T**	**F**
(4)	**F**	**T**	**T**	**F**	**U**
(5)	**F**	**T**	**T**	**F**	**T**
(6)	**T**	**F**	**T**	**F**	**U**
(7)	**T**	**F**	**T**	**F**	**T**

Row (ID)	p	$\neg p$	$p \vee \neg p$	$\neg(p \vee \neg p)$	$\neg\neg(p \vee \neg p)$
(1)	$\times$	6, 7	$\times$	4, 5, 6, 7	3
(2)	$\times$	7	$\times$	5, 7	$\times$
(3)	$\times$	$\varnothing$	$\times$	$\times$	$\times$
(4)	$\times$	$\times$	$\times$	$\times$	$\varnothing$
(5)	$\times$	$\times$	$\times$	$\times$	$\times$
(6)	$\times$	$\times$	$\times$	$\times$	$\varnothing$
(7)	$\times$	$\times$	$\times$	$\times$	$\times$

Figure: Reductions

- $\varnothing$ $v(\alpha) = \mathbf{U}$ and $\neg\exists w$ s.t $w(\alpha) = \mathbf{F}$
- $\times$ $v(\alpha) \neq \mathbf{U}$

Figure: Caption for reductions in some cell $\langle v, \alpha \rangle$.

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(7)	**T**	**F**	**T**	**F**	**T**

Figure: Second table

Row (ID)	p	$\neg p$	$p \vee \neg p$	$\neg(p \vee \neg p)$	$\neg\neg(p \vee \neg p)$
(1)	$\times$	7	$\times$	5, 7	$\emptyset$
(2)	$\times$	7	$\times$	5, 7	$\times$
(5)	$\times$	$\times$	$\times$	$\times$	$\times$
(7)	$\times$	$\times$	$\times$	$\times$	$\times$

Figure: Second cycle

Row (ID)	p	$\neg p$	$p \vee \neg p$	$\neg(p \vee \neg p)$	$\neg\neg(p \vee \neg p)$
(2)	**F**	**U**	**F**	**U**	**T**
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(7)	**T**	**F**	**T**	**F**	**T**

Figure: Final table

Summarizing

1. We have proposed a new decision procedure for Intuitionistic Propositional Logic.
2. The method was proven to be sound and complete
3. An upper bound for a formula α is $2^{|sub(\alpha)|}$
4. The time complexity of the procedure is $O(l^3 \times c^3)$

Thank you!

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