THE RING OF ISOBARIC POLYNOMIALS AND HYPERBOLIC GEOMETRY

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ABSTRACT: In this paper we first recall two important sequences in the ring of isobaric polynomials, generalized Fibonacci polynomials and generalized Lucas polynomials. Then we define isobaric logarithm and isobaric exponential operators and show how a hyperbolic trigonometry is constructed in the ring of isobaric polynomials.

Keywords: generalized Fibonacci polynomials, generalized Lucas polynomials, isobaric logarithm operator, isobaric exponential operator, isobaric hyperbolic trigonometric functions.

1. INTRODUCTION

An isobaric polynomial is a symmetric polynomial written on the elementary symmetric polynomial basis [2]. These polynomials have the pleasant property of being indexed by partitions of the natural numbers, and they turn out to provide useful representations of several arithmetic and algebraic structures [1, 3]. We introduce two new operators, the *isobaric logarithm* and the *isobaric exponential* operator from which we can define the *isobaric hyperbolic trigonometric functions*: the isobaric hyperbolic sine, cosine, tangent, etc. for which all of the usual hyperbolic trigonometric identities are satisfied. Since the ring of isobaric polynomials is isomorphic to the ring of symmetric polynomials, all of these identities induce (new) identities in the ring of symmetric polynomials, and imply the existence of a hyperbolic geometric structure for the ring of symmetric polynomials.

We are dealing with an algebra (or rather two isomorphic algebras), which has the property of supporting a hyperbolic geometry. This suggests the question: which linear algebras are capable of supporting a geometric structure, a question similar to the one answered by the Geometrization Theorem due to William Thurston, in which it is proved that every 3-dimensional manifold suitably restricted supports at least one of eight kinds of geometries. The results of this paper for our question show that for a positive answer, it is sufficient that such algebra have an invertible logarithmic operator.

In Section 2, we recall two important sequences, generalized Fibonacci polynomials and generalized Lucas polynomials. In Section 3, we define isobaric logarithm and isobaric exponential operators. Using these two isobaric operators we introduce isobaric hyperbolic trigonometric functions in Section 4. And conclusions are given in Section 5.

2. TWO IMPORTANT SEQUENCES

The ring of isobaric polynomials has two especially interesting bases directly related to arithmetic number theory; namely, the set of generalized Fibonacci polynomials (GFP) and the set of generalized Lucas polynomials (GLP) [2]. They are simply the isobaric images of the complete (or homogeneous) polynomials, and the isobaric images of the power sum symmetric polynomials, which are known to be bases of the ring of symmetric polynomials [2]. We can represent these two sequences explicitly as follows

Let $F_{k,0} = 1$ and $G_{k,0} = k$. Then

$$F_{k,n} = \sum_{\alpha \vdash n} \binom{|\alpha|}{\alpha_1, \ldots, \alpha_k} t_1^{\alpha_1} t_2^{\alpha_2} \cdots t_k^{\alpha_k},$$

and

$$G_{k,n} = \sum_{\alpha \vdash n} \binom{|\alpha|}{\alpha_1, \ldots, \alpha_k} \frac{n}{|\alpha|} t_1^{\alpha_1} t_2^{\alpha_2} \cdots t_k^{\alpha_k}.$$

These two sequences can also be defined by linear recursions:

$$F_{k,n} = t_1 F_{k,n-1} + t_2 F_{k,n-2} + \cdots + t_k F_{k,n-k},$$

and

$$G_{k,n} = t_1 G_{k,n-1} + t_2 G_{k,n-2} + \cdots + t_k G_{k,n-k}.$$

Hence the recursions can be extended to negative indices as well. The sequences can also be produced in another useful way which quickly leads to further applications.

We call the polynomial

$$C(X) = X^k - t_1 X^{k-1} - \cdots - t_k =: [t_1, t_2, \ldots, t_k]$$

the *core Polynomial*. We consider the companion matrix for the core polynomial $C(X)$, namely, the $k \times k$-matrix

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ t_k & t_{k-1} & \cdots & t_2 & t_1 \end{pmatrix}.$$

We let the companion matrix operate on its last row vector on the right, and append the image vector to the companion matrix as a new last row. We repeat this process obtaining a matrix with infinitely many rows. Assuming that $t_k \neq 0$ we can also extend the matrix from the top row upward by operating on the top row with A^{-1}, getting a doubly infinite matrix, he *infinite companion matrix* A^{∞}. The right hand column of A^{∞} is the sequence GFP, while the traces of the $k \times k$ contiguous blocks of A^{∞} are the elements of GLP.

The isobaric ring is also provided with a useful (commutative, associative) *convolution product* [2]:

$$P_{k,n} * P'_{k,n} = \sum P_{k,j} P'_{k,n-j}.$$

MacHenry and Wong showed that the group of multiplicative arithmetic functions (MF) in the ring of arithmetic functions (AF), can be faithfully represented by the GLP, where the convolution product in the ring of isobaric polynomials represents the Dirichlet product in the ring AF, and the group of additive arithmetic functions (AddF) is faithfully represented by GLP, again using the convolution product in the isobaric ring [3]. These representations have been useful in solving problems in the theory of arithmetic functions, for example, in embedding MF in its divisible closure [3].

3. THE ISOBARIC LOGARITHM AND EXPONENTIAL OPERATORS

The *isobaric logarithm* of an isobaric polynomial P by $L(P)$, and with a set of recursion parameters $\{t_1, t_2, \ldots, t_k\}$, is defined by

$$L(P_n) = \sum_{j=1}^{n-1} -j t_{n-j} P_j + k P_n.$$

This operator has the following matrix representation:

$$M_n(L) = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ -t_1 & 2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -t_{n-1} & -2t_{n-2} & \cdots & n \end{pmatrix},$$

which is invertible with respect to convolution product:

$$M_n(L)^{-1} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ \frac{1}{2}F_1 & \frac{1}{2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{n}F_{n-1} & \frac{1}{n}F_{n-2} & \cdots & \frac{1}{n} \end{pmatrix}.$$

Define $E_L = M_n(L)^{-1}$, which can also be written as the *isobaric exponential* operator E:

$$E(P_n) = \frac{1}{n}\sum_{j=1}^{n} F_{k,n-j}P_{k,j}.$$

Define $E(G_0) = 1$. Then in particular, $E(G_n) = F_n$.

Let $C(X) = [t_1, t_2, \ldots, t_k]$ and $C'(X) = [t'_1, t'_2, \ldots, t'_k]$.

Theorem 1. $L(F_{k,n} * F'_{k,n}) = L(F_{k,n}) + L(F'_{k,n})$ and $E(G_{k,n} + G'_{k,n}) = E(G_{k,n}) * E(G'_{k,n})$.

Theorem 2. $L(F_n^{*r}) = rL(F_n)$ and $E(rG_n) = E(G_n)^{*r}$.

These theorems have implications for arithmetic function theory. They also allow us to imitate the analytic definition of certain trigonometric functions in using our exponential operator.

4. THE ISOBARIC HYPERBOLIC TRIGONOMETRIC FUNCTIONS

We will define the hyperbolic trigonometric functions in terms of the exponential operator as is frequently done in analysis on arguments coming from the sequence of generalized Lucas polynomials, that is, on the set $\{G_{k,n}\}$, using the notation $C(G)$ and $S(G)$, respectively, for the isobaric hyperbolic cosine and the isobaric hyperbolic sine. The remaining four hyperbolic functions are defined in terms of these two in the usual way.

Set $G = G_{k,n}$, then define the *isobaric hyperbolic cosine* $C(G)$ and the *isobaric hyperbolic sine* $S(G)$ as follows:

$$C(G) = \frac{1}{2}(E(G) + E(G)^{-1})$$

and

$$S(G) = \frac{1}{2}(E(G) - E(G)^{-1}).$$

Since $E(G_n) = F_n$ and $F_n^{-1} = -\mathbf{t}_n$, we have the result:

Proposition 1. $C(G_n) = \frac{1}{2}(E(G_n) - \mathbf{t}_n)$ and $S(G) = \frac{1}{2}(E(G) + \mathbf{t}_n)$.

Let δ be the sequence $(1, 0, 0, \ldots, 0, \ldots)$.

Theorem 3. $C(G)^{*2} - S(G)^{*2} = \delta$.

Theorem 4. Let G and G' be two GLP induced by the core polynomials $[t_1, t_2, \ldots, t_k]$ and $[t'_1, t'_2, \ldots, t'_k]$, respectively, with $L(F) = G$ and $L(F') = G'$. Then

$$C(G + G') = C(G) * C(G') + S(G) * S(G')$$

and

$$S(G + G') = S(G) * C(G') + C(G) * S(G').$$

The proofs can be found in [1].

5. CONCLUSIONS

The ring of isobaric polynomials, being an isomorphic copy of the ring of symmetric polynomials, is very useful as its applications to the ring (UFD) of arithmetic functions illustrates. It was shown in [3], for example, that there are subgroups in the ring of isobaric polynomials that are isomorphic to the subgroups of Multiplicative Arithmetic Functions and the subgroup of Additive arithmetic functions, in the ring of arithmetic functions. It turns out using the algebra of isobaric functions one can construct the machinery for proving interesting theorems about these subgroups that transfer through the isomorphism [3]. In particular, the new operators and their relations in this paper give rise to new identities in both the ring of symmetric polynomials and in the ring of arithmetic functions. Moreover, the fact that the isobaric ring "supports" a hyperbolic geometry is also carried over to these other rings through an appropriate isomorphism. In addition to these consequences, we have the interesting problem: which linear algebras support, say, a hyperbolic geometry, or other of the interesting classical geometries, a problem analogous to the one suggested by the Geometrization Conjecture (now a theorem) proposed by William Thurston [4].

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